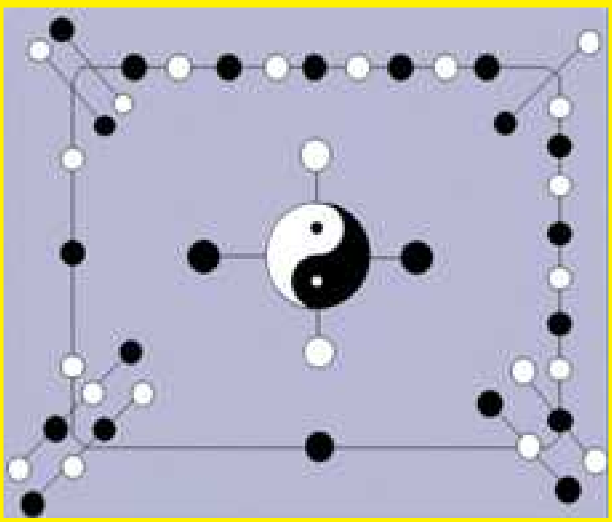
ISBN 978-1-59973-377-7

VOLUME 3, 2015

MATHEMATICAL COMBINATORICS

(INTERNATIONAL BOOK SERIES)

Edited By Linfan MAO



THE MADIS OF CHINESE ACADEMY OF SCIENCES AND

ACADEMY OF MATHEMATICAL COMBINATORICS & APPLICATIONS

September, 2015

Vol.3, 2015 ISBN 978-1-59973-377-7

MATHEMATICAL COMBINATORICS

(INTERNATIONAL BOOK SERIES)

Edited By Linfan MAO

The Madis of Chinese Academy of Sciences and

Academy of Mathematical Combinatorics & Applications

September, 2015

Aims and Scope: The **Mathematical Combinatorics (International Book Series)** is a fully refereed international book series with ISBN number on each issue, sponsored by the *MADIS of Chinese Academy of Sciences* and published in USA quarterly comprising 100-150 pages approx. per volume, which publishes original research papers and survey articles in all aspects of Smarandache multi-spaces, Smarandache geometries, mathematical combinatorics, non-euclidean geometry and topology and their applications to other sciences. Topics in detail to be covered are:

Smarandache multi-spaces with applications to other sciences, such as those of algebraic multi-systems, multi-metric spaces,$\cdots$, etc.. Smarandache geometries;

Topological graphs; Algebraic graphs; Random graphs; Combinatorial maps; Graph and map enumeration; Combinatorial designs; Combinatorial enumeration;

Differential Geometry; Geometry on manifolds; Low Dimensional Topology; Differential Topology; Topology of Manifolds; Geometrical aspects of Mathematical Physics and Relations with Manifold Topology;

Applications of Smarandache multi-spaces to theoretical physics; Applications of Combinatorics to mathematics and theoretical physics; Mathematical theory on gravitational fields; Mathematical theory on parallel universes; Other applications of Smarandache multi-space and combinatorics.

Generally, papers on mathematics with its applications not including in above topics are also welcome.

It is also available from the below international databases:

Serials Group/Editorial Department of EBSCO Publishing
10 Estes St. Ipswich, MA 01938-2106, USA
Tel.: (978) 356-6500, Ext. 2262 Fax: (978) 356-9371
http://www.ebsco.com/home/printsubs/priceproj.asp

and

Gale Directory of Publications and Broadcast Media, Gale, a part of Cengage Learning
27500 Drake Rd. Farmington Hills, MI 48331-3535, USA
Tel.: (248) 699-4253, ext. 1326; 1-800-347-GALE Fax: (248) 699-8075
http://www.gale.com

Indexing and Reviews: Mathematical Reviews (USA), Zentralblatt Math (Germany), Referativnyi Zhurnal (Russia), Mathematika (Russia), Directory of Open Access (DoAJ), International Statistical Institute (ISI), International Scientific Indexing (ISI, impact factor 1.416), Institute for Scientific Information (PA, USA), Library of Congress Subject Headings (USA).

Subscription A subscription can be ordered by an email directly to

Linfan Mao
The Editor-in-Chief of *International Journal of Mathematical Combinatorics*
Chinese Academy of Mathematics and System Science
Beijing, 100190, P.R.China
Email: maolinfan@163.com

Price: US$48.00

Famous Words:

It is at our mother's knee that we acquire our noblest and truest and highest ideals, but there is seldom any money in them.

By Mark Twain, an American writer.

Math.Combin.Book Ser. Vol.3(2015), 01-32

A Calculus and Algebra Derived from Directed Graph Algebras

Kh.Shahbazpour and Mahdihe Nouri

(Department of Mathematics, University of Urmia, Urmia, I.R.Iran, P.O.BOX, 57135-165)

E-mail: shahbazpour@hotmail.com

Abstract: Shallon invented a means of deriving algebras from graphs, yielding numerous examples of so-called graph algebras with interesting equational properties. Here we study directed graph algebras, derived from directed graphs in the same way that Shallon's undirected graph algebras are derived from graphs. Also we will define a new map, that obtained by Cartesian product of two simple graphs p_n, that we will say from now the mah-graph. Next we will discuss algebraic operations on mah-graphs. Finally we suggest a new algebra, the mah-graph algebra (Mah-Algebra), which is derived from directed graph algebras.

Key Words: Direct product, directed graph, Mah-graph, Shallon algebra, kM-algebra

AMS(2010): 08B15, 08B05.

§1. Introduction

Graph theory is one of the most practical branches in mathematics. This branch of mathematics has a lot of use in other fields of studies and engineering, and has competency in solving lots of problems in mathematics. The Cartesian product of two graphs are mentioned in[18]. We can define a graph plane with the use of mentioned product, that can be considered as isomorphic with the plane $\mathbb{Z}^+ * \mathbb{Z}^+$.

Our basic idea is originated from the nature. Rivers of one area always acts as unilateral courses and at last, they finished in the sea/ocean with different sources. All blood vessels from different part of the body flew to heart of beings. The air-lines that took off from different part of world and all landed in the same airport. The staff of an office that worked out of home and go to the same place, called Office, and lots of other examples give us a new idea of directed graphs. If we consider directed graphs with one or more primary point and just one conclusive point, in fact we could define new shape of structures.

A km-map would be defined on a graph plane, made from Cartesian product of two simple graphs $p_n * p_n$. The purpose of this paper is to define the kh-graph and study of a new structure that could be mentioned with the definition of operations on these maps.

The km-graph could be used in the computer logic, hardware construction in smaller size with higher speed, in debate of crowded terminals, traffics and automations. Finally by rewriting

[1]Received January 1, 2015, Accepted August 6, 2015.

the km-graphs into mathematical formulas and identities, we will have interesting structures similar to Shallon's algebra (graph algebra).

At the first part of paper, we will study some preliminary and essential definitions. In section 3 directed graphs and directed graph algebras are studied. The graph plane and mentioned km-graph and its different planes and structures would be studied in section 4.

§2. Basic Definitions and Structures

In this section we provide the basic definitions and theorems for some of the basic structures and ideas that we shall use in the pages ahead. For more details, see [18], [23], [12].

Let A be a set and n be a positive integer. We define A^n to be the set of all n-tuples of A, and $A^0 = \emptyset$. The natural number n is called the *rank* of the operation, if we call a map $\phi : A^n \to A$ an $n-ary$ *operation* on A. Operations of rank 1 and 2 are usually called unary and binary operations, respectively. Also for all intends and purposes, nullary operations (as those of rank 0 are often called) are just the elements of A. They are frequently called constants.

Definition 2.1 *An algebra is a pair $< A, F >$ in which A is a nonempty set and $F =< f_i : i \in I >$ is a sequence of operations on A, indexed by some set I. The set A is the universe of the algebra, and the f_i's are the fundamental or basic operations.*

For our present discussion, we will limit ourselves to finite algebras, that is, those whose universes are sets of finite cardinality . The *equational theory* of an algebra is the set consisting of all equations true in that algebra. In the case of groups, one such equation might be the associative identity. If there is a finite list of equations true in an algebra from which all equations true in the algebra can be deduced, we say the algebra is *finitely based*. For example in the class of one-element algebras, each of those is finitely based and the base is simply the equation $x \approx y$. We typically write A to indicate the algebra $< A, F >$ expect when doing so causes confusion. For each algebra A, we define a map $\rho : I \to \omega$ by letting $\rho(i) = rank(F_i)$ for every $i \in I$. The set I is called the *set of operation symbols*. The map ρ is known as the *signature* of the algebra A and it simply assigns to each operation symbol the natural number which is its rank. When a set of algebras share the same signature, we say that they are *similar* or simply state their shared signature.

If κ is a class of similar algebras, we will use the following notations:

$H(\kappa)$ represents the class of all homomorphic images of members of κ;

$S(\kappa)$ represents the class of all isomorphic images of sub algebras of members of κ;

$P(\kappa)$ represents the class of all direct products of system of algebras belonging to κ.

Definition 2.2 *The class ν of similar algebras is a variety provided it is closed with respect to the formation of homomorphic images, sub algebras and direct products.*

According to a result of Birkhoff (Theorem 2.1), it turns out that ν is a variety precisely if it is of the form $HSP(\kappa)$ for some class κ of similar algebras. We use $HSP(A)$ to denote the

variety generated by an algebra A. The equational theory of an algebra is the set consisting of all equations true in that algebra. In order to introduce the notion of equational theory, we begin by defining the set of terms.

Let $T(X)$ be the set of all terms over the alphabet $X = \{x_0, x_1, \cdots\}$ using juxtaposition and the symbol ∞. $T(X)$ is defined inductively as follows:

(i) every $x_i, (i = 0, 1, 2, \cdots)$ (also called variables) and ∞ is a term;

(ii) if t and t' are terms, then (tt') is a term;

(iii) $T(X)$ is the set of all terms which can be obtained from (i) and (ii) in finitely many steps.

The *left most variable* of a term t is denoted by $Left(t)$. A term in which the symbol ∞ occurs is called trivial. Let $T'(X)$ be the set of all non-trivial terms. To every non-trivial term t we assign a directed graph $G(t) = (V(t), R(t))$ where $V(t)$ is the set of all variables and $R(t)$ is defined inductively by $R(t) = \emptyset$ if $t \in X$ and $R(tt') = R(t) \cup R(t') \cup \{Left(t), Left(t')\}$. Note that $G(t)$ always a connected graph.

An *equation* is just an ordered pair of terms. We will denote the equation (s, t) by $s \approx t$. We say an equation $s \approx t$ is true in an algebra A provided s and t have the same signature. In this case, we also say that A is a model of $s \approx t$, which we will denote by $A \models s \approx t$.

Let κ be a class of similar algebras and let Σ be any set of equations of the same similarity type as κ. We say that κ is a class of models Σ(or that Σ is true in κ) provided $A \models s \approx t$) for all algebras A found in κ and for all equations $s \approx t$ found in Σ. We use $\kappa \models \Sigma$ to denote this, and we use $Mod\Sigma$ to denote the class of all models of Σ.

The set of all equations true in a variety ν (or an algebra A) is known as the *equational theory* of ν (respectively, A). If Σ is a set of equations from which we can derive the equation $s \approx t$, we write $\Sigma \vdash s \approx t$ and we say $s \approx t$ is *derivable* from Σ. In 1935, Garrett Birkhoff proved the following theorem:

Theorem 2.1 (Bikhoffs HSP Theorem) *Let ν be a class of similar algebras. Then ν is a variety if and only if there is a set σ of equations and a class κ of similar algebras so that $\nu = HSP(\kappa) = Mod\Sigma$.*

From this theorem, we have a clear link between the algebraic structures of the variety ν and its equational theory.

Definition 2.3 *A set Σ of equations is a base for the variety ν (respectively, the algebra A) provided ν (respectively, HSP(A)) is the class of all models of* Σ.

Thus an algebra A is finitely based provided there exists a finite set Σ of equations such that any equation true in A can be derived from Σ. That is, if $A \models s \approx t$ and Σ is a finite base of the equational theory of A, then $\Sigma \vdash s \approx t$. If a variety or an algebra does not have a finite base, we say that it fails to finitely based and we call it *non-finitely based*.

We say an algebra is *locally finite* provided each of its finitely generated sub algebras is finite, and we say a variety is locally finite if each of its algebras is locally finite.

A useful fact is the following:

Theorem 2.2 *Every variety generated by a finite algebra is locally finite.*

Thus if **A** is an inherently nonfinitely based finite algebra that is a subset of **B**, where **B** is also finite, then **B** must also be inherently nonfinitely based. It is in this way that the property of being inherently nonfinitely based is contagious. In an analogous way, we define an inherently non-finitely based variety ϑ as one in which the following conditions occur:

(i) ϑ has a finite signature;
(ii) ϑ is locally finite;
(iii) ϑ is not included in any finitely based locally finite variety.

Let ν be a variety and let $n \in \omega$. The class $\nu^{(n)}$ of algebras is defined by the following condition:

An algebra B is found in $\nu^{(n)}$ if and only if every sub algebra of B with n or fewer generators belongs to ν. Equivalently, we might think of $\nu^{(n)}$ as the variety defined by the equations true in $\nu(n)$ that have n or fewer variables. Notice that $(\nu \subseteq \cdots \subseteq \nu^{(n+1)} \subseteq \nu^{(n)} \subseteq \nu^{(n-1)} \subseteq \cdots)$ and $\nu = \bigcap_{n\in\omega}^{nu^{(n)}}$.

A nonfinitely based algebra A might be nonfinitely based in a more infectious manner:

It might turn out that if A is found in $HSP(B)$, where B is a finite algebra, then B is also nonfinitely based. This leads us to a stronger non finite basis concept.

Definition 2.4 *An algebra A is inherently non-finitely based provided:*

(i) A has only finitely many basic operations;
(ii) A belongs to some locally finite operations;
(iii) A belong to no locally finite variety which is finitely based.

In an analogous way, we say that a locally finite variety v of finite signature is inherently nonfinitely based provided it is not included in any finitely based locally finite variety. In [2], Birkhoff observed that $v^{(n)}$ is finitely based whenever v is a locally finite variety of finite signature. As an example, let v be a locally finite variety of finite signature. If the only basic operations of v are either of rank 0 or rank 1, then every equation true in v can have at most two variables. In other words, $v = v^{(2)}$ and so v is finitely based. From Birkhoff's observation, we have the following as pointed out by McNulty in [15]:

Theorem 2.3 *Let v be locally finite variety with a finite signature. Then the following conditions are equivalent:*

(i) v is inherently non-finitely based;
(ii) The variety $v^{(n)}$ is not locally finite for any natural number n;
(iii) For arbitrary large natural numbers N, there exists a non-locally finite algebra B_N whose N-generated sub algebras belong to v.

Thus to show that a locally finite variety v of finite signature is inherently nonfinitely based, it is enough to construct a family of algebras B_n (for each $n \in \omega$) so that each B_n fails to be locally finite and is found inside $v^{(n)}$.

In 1995, Jezek and McNulty produced a five element commutative directoid and showed that while it is nonfinitely based, it fails to be inherently based [17]. This resolved the original question of jezek and Quackenbush but did not answer the following:

Is there a finite commutative directed that is inherently non-finitely based?

In 1996, E.Hajilarov produced a six-element commutative directoid and asserted that it is inherently based [5]. We will discuss an unresolved issue about Hajilarov's directoid that reopens its finite basis question. We also provide a partial answer to the modified question of jezek and Quackenbush by constructing a locally finite variety of commutative directoids that is inherently nonfinitely based.

A *sub direct representation* of an algebra $\mathbf{A}$ is a system $< h_i : i \in I >$ of homomorphisms, all with domain A, that separates the points of A: that is, if a and b are distinct elements of A, ten there is at least one $i \in I$ so that $h_i(a) \neq h_i(b)$. The algebra $h_i(A)$ are called **subdirectfactors** of the representation. Starting with a complicated algebra $\mathbf{A}$, one way to better understand its structure is to analyze a system $< h_i(A)\ i \in I >$ of potentially less complicated homomorphic image, and a sub direct representation of $\mathbf{A}$ provides such a system.

The *residual bound* of a variety ϑ is the least cardinal κ (should one exist) such that for every algebra $\mathbf{A} \in \vartheta$ there is a sub direct representation $< h_i i \in I >$ of $\mathbf{A}$ such that each sub direct factor has fewer than κ elements. If a variety ν of finite signature has a finite residual bound, it also satisfies the following condition: there is a finite set $\mathbf{S}$ of finite algebras belonging to ϑ so that every algebra in ϑ has a sub direct representation using only sub direct factors from $\mathbf{s}$.

According to a result of Robert Quackenbush, if a variety generated by a finite algebra has an infinite sub directedly irreducible member, it must also have arbitrarily large finite once [20]. In 1981, Wieslaw Dziobiak improved this result by showing that the same holds in any locally finite variety [3]. A Problem of Quackenbush asks whether there exists a finite algebra such that the variety it generates contains infinitely many distinct (up to isomorphism) sub directly irreducible members but no infinite ones. One of the thing Ralph McKenzie did in [13] was to provide an example of 4-element algebra of countable signature that generates a variety with this property. Whether or not this is possible with an algebra with only finitely many basic operations is not yet know.

If all of the sub directly irreducible algebra in a variety are finite, we say that the variety is *residual finite*. Starting with a finite algebra, there is no guarantee that the variety it generates is residually finite, nor that the algebra is finitely based. The relationship between these three finiteness conditions led to the posing of the following problem in 1976:

Is every finite algebra of finite signature that generates a variety with a finite a finite residual bound finitely based?

Bjarni Jonsson posed this problem at a meeting at a meeting at the Mathematical Research Institute in Oberwolfach while Robert Park offered it as a conjecture in his PH.D. dissertation [19]. At the time this problem was framed, essentially only five nonfinitely based finite algebras were known. Park established that none of these five algebras could be a counterexample. In

the ensuing years our supply of nonfinitely based finite algebras has become infinite and varied, yet no counterexample is known. Indeed, Ros Willard [26] has offered a 50 euro reward for the first published example of such an algebra. In chapter 2, we show that a wide class of algebras known to be nonfinitely based will not supply such an example. It is still an open problem whether some of the nonfinitely based finite algebras known today generate varieties with finite residual bound. while the condition of generating a variety with a finite residual bound could well be sufficient to ensure that a finite algebra is finitely based, it is in its own right a very subtle property of finite algebras. Indeed, Ralph McKenzie has shown that there is no algorithm for recognizing when a finite algebra has this property [13]. The last question motivating our research was originally formulated in 1976 by Eilenberg and Schutzenberger [4]. In their investigation of pseudo varieties, they ask, If ϑ is a variety generated by a finite algebra, W is a finitely based variety, and ϑ and W share the same finite algebras, must ϑ be finitely based?

To answer this question in the negative, one would need to supply a finite, nonfinitely based algebra to generate ϑ and a finitely based variety W so that ϑ and W have the same finite algebras. McNulty, Szekely, and Willard have proven that no counterexample can be found among a wide class of finite, non finitely based algebras [26]; furthermore they noted that this property also cannot be recognized by any algorithm.We show that the locally finite, inherently based variety of commutative directoids we construct will fail to yield a counterexample if it is shown to be generated by a finite algebra.

§3. Directed Graphs and Directed Graph Algebras

Definition 3.1 *A graph G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each two vertexes called its conclusive points.*

Definition 3.2 *A path in the directed graph (v, E) is an ordered $(n+1)-tuple$ $(x_1, \cdots, x_{n+1})$ such that $(x_i, x_{i+1}) \in E$ for all $i = 1, \cdots, n$. The path $(x_1, ..., x_{n+1})$ has length n. A cycle is a path from some vertex to itself. Given a graph G and $x \in V_G$, $[x\rangle_G$, (or just $[x\rangle$, when G is clear), is the set of $y \in V_G$ such that there is a path from x to y in G. A directed graph is acyclic if it contains no cycles. A directed graph (V, E) is loop-free if there is no cycle of length 1, that is if there is no $x \in V$ such that $(x, x) \in E$. A directed graph (V, E) is looped if there is a loop at every vertex, that is if $(x, x) \in E$ for every $x \in V$.*

Definition 3.3 *A directed graph or digraph $G =< V, E >$ is a triple consisting of a nonempty set $V(G)$ of elements called vertexes, together with a set $E(G)$ of ordered pairs from $V \times V \to V$, called edges, and a map that assigning to each edge an ordered pair of vertexes.*

Thus our directed graphs do not allow multiple edges, but they do allow edges of the form (x, x) (that is, we allow vertexes to be looped). Given a directed graph G, we can refer to the vertex set and edge set of G as V_G and E_G, respectively. Let us say that $\acute{G}$ is a subgraph of G if $V_{\acute{G}} \subseteq V_G$ and $E_{\acute{G}} = E_G \bigcap (V_{\acute{G}} \times V_{\acute{G}})$. When we draw a directed graph, we generally draw the edge (x, y) as an arrow from vertex x to y. When drawing an undirected graph, we simply

draw the edge (x, y) as a line from x to y; since we know that (y, x) must also be in the graph edge set, there is no question of which way the edge goes. Let us call any variety generated by directed graph algebras a directed graph variety. As noted in [25], any directed graph variety ν contains $A(G)$ for all G that are direct products, subgraphs, disjoint unions, directed unions and homomorphic images of directed graphs underlying algebra in ν. More generally, any variety is closed under homomorphisms, sub algebras and direct products. We shall need this more general fact to obtain the results in section. The following definition is from [25].

Definition 3.4([25]) *Let $G = (V, E)$ be a directed graph. The directed graph algebra $A(G)$ is the algebra with underlying set $\{v \bigcup \infty\}$, where $\infty \notin V$, and two basic operations: one nullary operation, also denoted by ∞, which has value ∞, and one binary operation, sometimes called multiplication, denoted by juxtaposition, which is given by*

$$(u, v) = \begin{cases} u & if (u, v) \in E \\ \infty & otherwise \end{cases}$$

Let $G =< V, E >$ be a complete undirected graph. We define a tournament T as an algebra with universe $V \cup \{\infty\}$ (where $\infty \notin V$) and binary relation $\twoheadrightarrow$ so that for distinct $x, y \in V$ exactly one of $x \twoheadrightarrow y$ and $y \twoheadrightarrow x$ is true. We make each edge directed using the relation $\twoheadrightarrow$, that is, we make the edge between x and y directed toward y if and only if $x \twoheadrightarrow y$. We can then make $\twoheadrightarrow$ into a binary operation as follows:

$$x.y = y.x = \begin{cases} x & if x \twoheadrightarrow y \\ \infty & otherwise \end{cases}$$

The relation $x \twoheadrightarrow y$ is generally read as "x defeats y" or " y loses to x" in the tournament T. Notice that in tournament we require a directed edge between any two distinct vertexes. When we draw a tournament, we will represent the ordered edge between any two distinct vertexes. When we draw a tournament, we will represent the ordered pair $(x, y) \in\twoheadrightarrow$ as vertexes joined by a double-headed arrow pointing away from $x.y$.

A semi-tournament T is simply a tournament in which we relax the restriction that the underlying graph G be complete. In this case, if x and y are distinct vertexes in v and neither $x \twoheadrightarrow y$ nor $y \twoheadrightarrow x$, we define $x.y = y.x = \infty$. In this way the element ∞ acts as a default element. The table of Park's semi-tournament P with three element is shown in Figure 2.

O	t	s	r	∞
t	t	t	∞	∞
s	t	s	s	∞
r	∞	s	r	∞
∞	∞	∞	∞	∞

Let v be the variety generated by p. We obtain graph algebras in exactly the same way,

except that in a graph algebra the underlying G is an undirected graph . one difference between our terminology and the terminology of the authors of [25] is that they refer to algebras $A(G)$ defined as graph algebras, while we refer to such algebras as directed graph algebra. Our purpose in doing so is to avoid confusion with the (undirected) graph algebra of [25]. Let $T(X)$ be the set of all terms over a set X of variables in the type of directed graph algebras. We shall make frequent use of the following definition and lemma from Kiss-poschel-prohle [25].

Definition 3.5([25]) *For $t \in T(X)$, the term graph $G(t) = (V(t), E(t))$ is the directed graph defined as follows. $v(t)$ is the set of variables that appear in t. $E(t)$ is defined inductively as follows:*

$E(t) = \phi$ if t is a variable, and $E(ts) = E(t) \cup E(S) \bigcup (L(t), L(s))$, where $L(t)$ is the leftmost variable that appears in t. The rooted graph derived from t is $(G(t), L(t))$.

As an example, consider the term $t = (x(y((zx)y)))x$, the term graph $G(t)$ is pictured in figure(1). Different terms can have the same term graph, another term that has the term graph pictured in Figure 1 is $(z((xx)y))(yz)$.

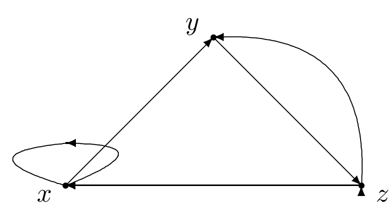


Figure 1 The term graph of $(x(y((zx)y)))x$

Following [25], we call a term trivial if ∞ occurs in it.

Lemma 3.1([25]) *Let $G = (V, E)$ be a graph , $t, s \in T(X)$, and $h : X \to A(G)$ an evaluation of the variable. Let the same h denote the unique extension of this evaluation to the algebra $T(x)$ of all terms.*

1. *If $t \in T(X)$ is nontrivial, then $(G(t), L(t))$ is a finite rooted graph. Conversely, for every finite rooted graph (G, ν) there exists $t \in T(V_G)$ with $G(t) = G$ and $L(t) = \nu$.*
2. *If t is a trivial term, or if h takes the value ∞ on some element of $V(t)$, then $h(t) = \infty$. Otherwise,if $h : G(t) \to G$ is a homomorphism of directed graphs, then $h(t) = h(L(t))$, and if h is not a homomorphism of directed graphs, then $h(t) = \infty$.*
3. *The identity $s \approx t$ is true in every graph algebra if and only if either both s and t are trivial terms, or neither of them is trivial, $G(s) = G(t)$, and $L(s) = L(t)$.*

Of course, when specifying an evaluation h of a particular term t, it is enough to define h on $V(t)$ rather than on all of X, and this is what we typically do. For efficiency, we typically say *graph homomorphism* when what we really mean is *directed graph homomorphism.*

A law $s \approx t$ is regular if $V(s) = V(t)$.

Lemma 3.2 *If D is a directed graph algebra that contains a loop, then all nontrivial laws of D are regular.*

Proof Suppose $s \approx t$ is a nontrivial law of D that is not regular, without loss of generality, we may take $x \in V(s) \setminus V(t)$. Let h be the graph homomorphism that maps x to ∞ and maps everything in $(V(S) \bigcup V(t)) \setminus x$ to some looped element a. Then $h(t)$ is a, but $h(s)$ is ∞, whence $s \approx t$ is not a law of D, contrary to our assumption.

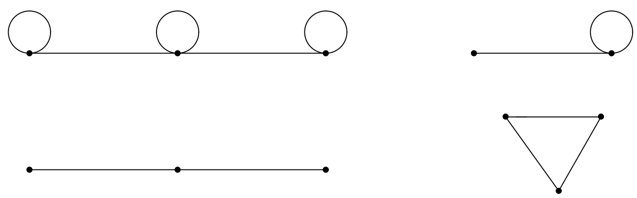


Figure 2 Graphs of the four minimal INFB graph algebras

In [1] it is shown that:

Definition 3.6 *A locally finite variety ν is finitely based, or FB, if there is a finite basis for the equations of ν. ν is inherently nonfinitely based, or $INFB$, if ν is not contained in any other FB locally finite variety. We say that the algebra A is FB if $\nu(A)$ is FB, which is the case exactly when there is a finite basis for $Eq(A)$. We say that A is INFB if ν is $INFB$. Note that if A is INFB then A is not FB; otherwise $\nu(A)$ would be contained in an FB locally finite variety, namely $\nu(A)$ itself.*

Theorem 3.1 *A graph algebra A is FB if and only if its underlying graph G_A has no subgraph isomorphic to one of the graphs in Figure 2*

This theorem gives a complete classification of the FB graph algebras. since every graph algebra is also a directed graph algebra, the above theorem will be of some use to us as we work to classify the FB directed algebras.

Our work falls under the heading of universal algebra, so we use the language and notation of that subject. Our algebras can be viewed as models in the sense of Model Theory, so we sometimes borrow from the notation of that subject as well. For example, we use $A \models \sigma$ to mean that the sentence σ is true in the algebra A, and we use $\Gamma \vdash \sigma$ to mean that there is a derivation of σ from the sentences in Γ.

We shall distinguish carefully between the symbols $=$ and $\approx$. We shall use $=$ only for exact equality; if we say , for example, $s = t$, then we mean that s and t are identical. We shall use $\approx$ when writing down laws. Thus we shall say things like $A \models s \approx t$ and $\Gamma \vdash s \models t$. (of course, it is true but uninteresting that $A \models s = s$ and $\Gamma \vdash s = s$ for every A, s, and Γ.)

When writing down a law λ, we shall use λ^L to refer to the term on the left-hand side of λ and λ^R to refer to the term on the right-hand side.

Given a term t, $l(t)$ is the length of t, defined by the following recursion:

$$l(t) = \begin{cases} 1 & if \text{ t } is\ a\ single\ variable \\ l(r) + l(s) & if\ t = rs. \end{cases}$$

Thus $l(t)$ is the number of places at which variables occur in t.

When dealing with terms, it is expeditious to avoid writing down as many parentheses as possible. Toward this end, we adopt the convention that sub terms will be grouped from the left. Thus, for example, when we say

$$xy_1y_2...y_n$$

we mean that

$$(\cdots((xy_1)y_2)\cdots)y_n.$$

When giving a derivation, we justify the steps as follows. When a step is justified by a numbered entity, such as an equation or proposition, the number appears underneath the $\approx$ or $=$ on that step's line in the proof. When the justification is something that does not have a number, the justification appears in square brackets at the right end of the line.

In general, we use u, v, w, x, y, and z for variables, s, t, and lowercase Greek letters for terms and sub terms, lower case Greek letters for laws, and uppercase Greek letters for sets of laws.

§4. **km-Graphs**

In this section we will define graph plane and then we will construct km-graphs. Next by using the language and notation of universal algebra, we will define a new algebra derived from directed graph algebras.

Definition 4.1 *Some of the structures with regular configurations can be expressed as the Cartesian product of two or more graphs. After the formation of the nodes of such a graph according to the nodes of the generators, a member should be added between two typical nodes* (u_i, v_j) *and* (u_k, v_l)*, as show in Figure 3, if the following conditions are satisfied*

$$[(u_i = u_k,\ v_l\ adj\ v_j)\ or\ (v_i = v_l,\ u_i\ adj\ u_k)].$$

Some other structures with regular configurations can be expressed as the strong Cartesian product of two or more graphs. After the formation of the nodes of such a graph a member should be added between two typical nodes (u_k, v_l) *and* (u_i, v_j) *if the following conditions are satisfied:*

$$[(u_i = u_k\ and\ v_l\ adj\ v_j)\ or\ (v_j = v_l,\ u_i\ adj\ u_k)]\ or\ u_i\ adj\ u_k\ and\ v_l\ adj\ v_j.$$

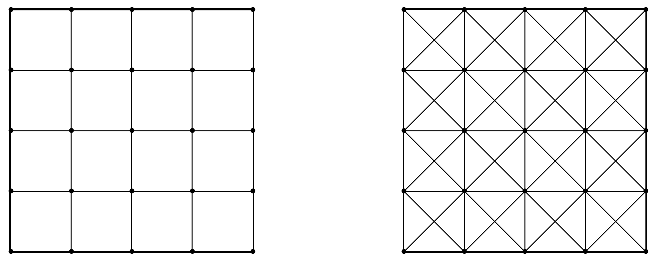


Figure 3 Cartesian product and strongly Cartesian product of p_n

Let p_n be a simple graph. By Cartesian product of $p_n * p_n$ we have a plane that from now we will call this plane the **graphic plane**. The graphic plane from right and up is infinite and from down and left is bounded. This plane can be considered isomorphic with the plane produced by $\mathbb{Z}^+ * \mathbb{Z}^+$, but in our plane the symmetric axis are defined in an other ways. Samples of graphic plane from natural are the chess plane, the factories producer line, train road of a country, air lines, electric cables in a city or home. The importance of our idea is to find simpler maps for relations between natural phenomena such that natural factors less often injured in connection together. First of all we will consider some axioms in our graphic plane:

Definition 4.2 *we will call the segment crossing from every node (u_i, v_i) for every $i \in N$ the symmetric axis of the graphic plane.*

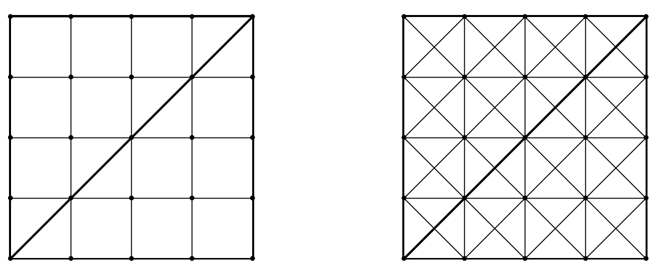


Figure 4 Symmetric axis of Cartesian product and strongly Cartesian product of $p_n \times p_n$

The symmetric axis splits the graphic plane into two half graphic plane that we will show the upper half graphic plane by positive and the lower half graphic plane by negative notations.

By the above definitions we will consider the nodes from the positive half graph plane with positive notations and the nodes from the negative half graph plane with negative notations. In this manner the all nodes on symmetric axis are without notations.

Definition 4.3 *A km-graph G_F consists of a vertex set $\{V(G_F) \bigcup \infty\}$ and an edge set $E(G_F)$,*

where for every $x, y \in V$.

$$G_F(x) = \begin{cases} y & if x \twoheadrightarrow y \\ \infty & otherwise \end{cases}$$

that satisfies the following conditions:

(i) There exist only one and unique conclusive point, that we will denote it by M*;*

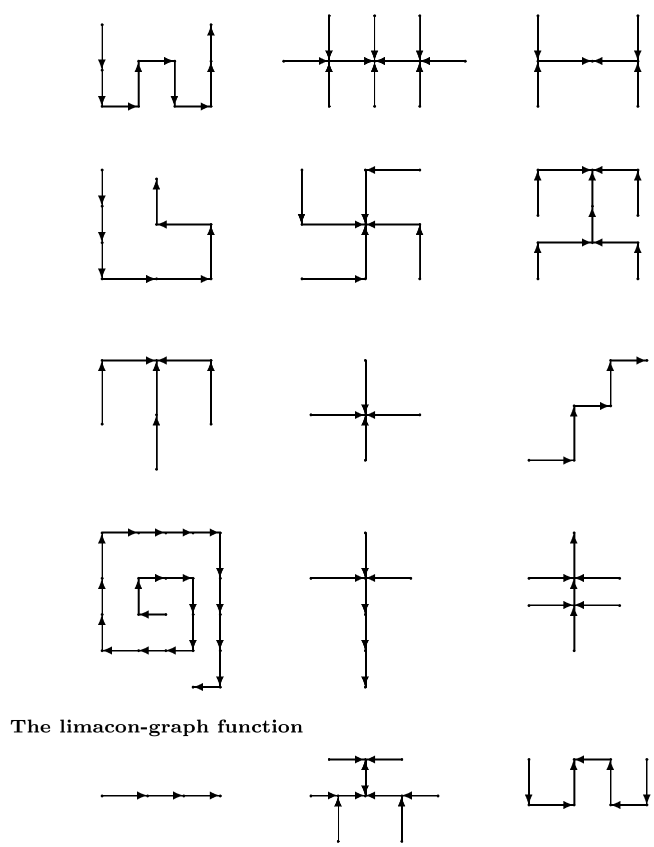
(ii) From every inception point in every path there is at least one tournament to M*. Therefore a km-graph may have many inception point;*

(iii) Each vertex is loop less;

(iv) There is no path of a vertex to itself. That is, there is no sequence of vertexes such that

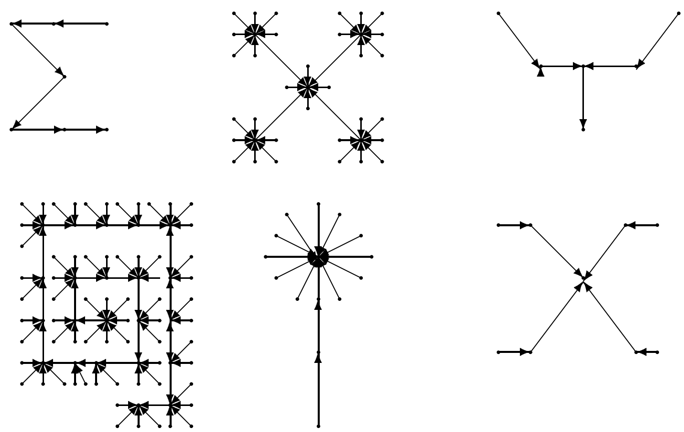
$$v_1 \longrightarrow v_2 \longrightarrow v_3 \longrightarrow \cdots \longrightarrow v_n \longrightarrow v_1.$$

Some of km-graphs are shown below:

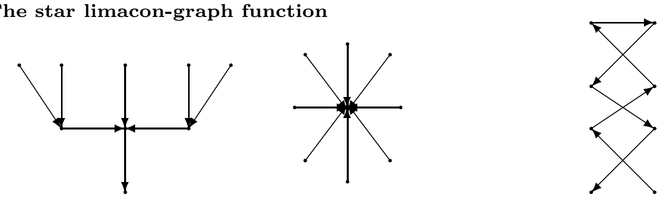


(a) Graph function in the graph plane

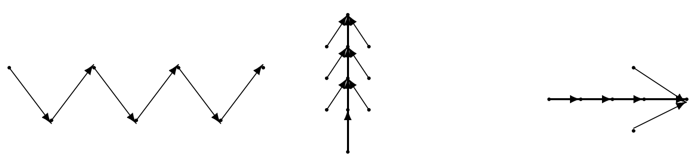
Figure 5 Samples of km-graphs



The star limacon-graph function

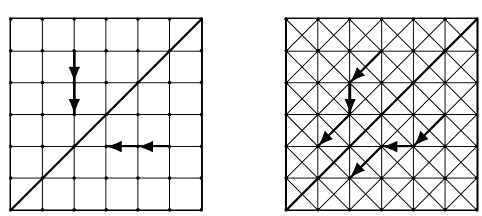


The star like-graph function



(b) Graph functions in the strong graph plane

Figure 6 Samples of km-graphs

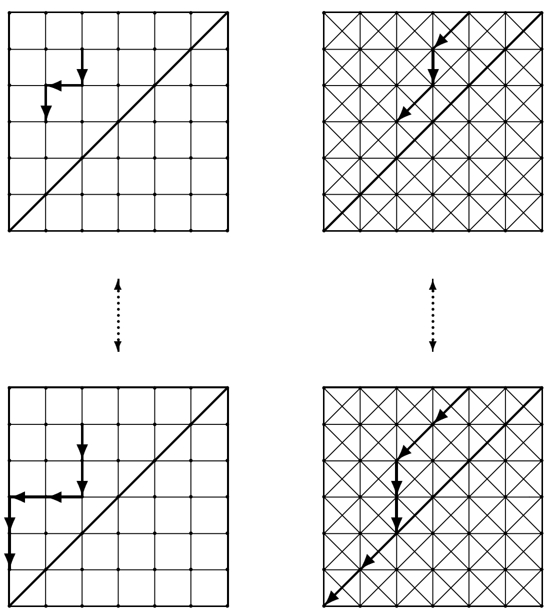


Symetry of a graph function

Figure 7 Sample of Symmetry for km-graphs

It is obvious that we can define the positive scalar multiplication on km-graphs. If let $k \in \mathbb{Z}$ be a integer number and $G_F(x)$ be a arbitrary km-graphs., then $kG_F(x)$ is a km-graph that every vertex of $kG_F(x)$ is k times as much of $G_F(x)$. Also one can consider inversee km-graph

of km-graph $G_F(x)$, denoted by $(G_F(x))^{-1}$, as a km-graph such that the direction of any vertex will be denoted by inversee direction. Therefore it is not true that the inversee of a km-graph is usually a km-graph. The only case that we have inversee km-graph is the km-graphs with only one inception point. On the other hand the symmetry of a km-graph $G_F(x)$ is the km-graph $G'_F(x)$ such that every vertex $v'_i \in G'_F(x)$ is symmetric by a vertex $v'_i \in G_F(x)$, with respect to symmetric axis of graph plane.



Scalar multiplication of a graph function

Figure 8 Sample for Scalar Multiplication of km-graphs for k = 2

Also we can define inverse km-graph as follows:

Definition 4.4 *The inversee of a km-graph G_F, that denoted by G_F^{-1}, obtained by change of path direction.*

Note that only km-graphs with one inception point and one conclusive point have inversee. That is, in general km-graphs don't have corresponding inversee km-graph.

Definition 4.5 *Two km-graphs G_{F_1} and G_{F_2} are equal if and only if they have the same vertex set and the same input set for a vertex. Although this a perfectly reasonable definition, for most purposes the module of relationship is not essentially changed if G_{F_2} is obtained from G_{F_1} just by renaming the vertex set.*

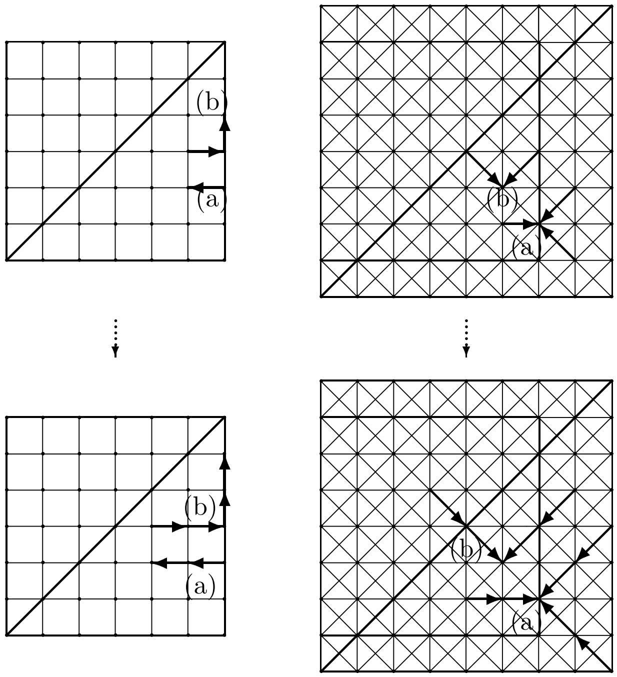


Figure 9 Samples for Scalar Multiplication km-graphs for K=2

For arbitrary two km-graphs G_{F_1} and G_{F_2}, we can define union and intersection of km-graphs.

Definition 4.6 *The union of two km-graph G_{F_1} and G_{F_2} is just obtained by superposition of conclusive points of G_{F_1} and G_{F_2}. There fore if G_{F_1} have V_{F_1} as vertex set and E_{F_1} as edge set and G_{F_2} have G_{F_1} have V_{F_2} as vertex set and E_{F_2} as edge set, then $G_{F_1} \cup G_{F_2}$ is defined by $V_{F_1} \cup V_{F_2}$ as vertex set and $E_{F_1} \cup E_{F_2}$ as edge set.*

It is obvious that union of two km-graph is not a km-graph in general. However we will define sum of two km-graph by similar definition without closed path in result. Before do it, in the following we have definition of intersection.

Definition 4.7 *Intersection of two km-graph G_{F_1} and G_{F_2} is just obtained by superposition of conclusive points of G_{F_1} and G_{F_2}, such that $G_{F_1} \cap G_{F_2}$ is defined by $V_{F_1} \cap V_{F_2}$ as vertex set and $E_{F_1} \cap E_{F_2}$ as edge set. The intersection of two km-graph may have two path that have similar vertexes but the edges are not in same direction. In this type the two edges will be deleted and we have the discrete km-graph.*

There fore we see that the intersection of two km-graph is not necessary to be a km-graph.

Definition 4.8 *A km-graph with at least one singular (isolated) vertex is called discrete km-graph. We will used the name $D - GF_i$ for discrete km-graphs.*

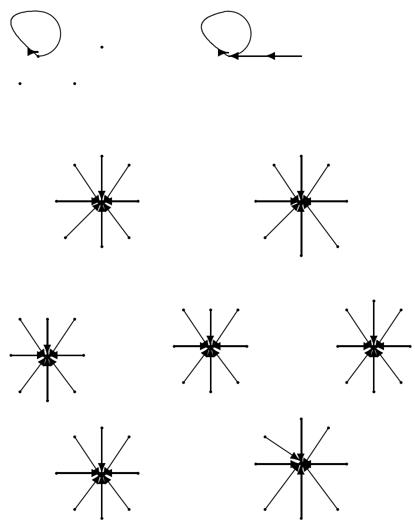


Figure 10 Sample of M_4,M_3,M_7-km-graph

Thus every single loop is a looped km-graph.

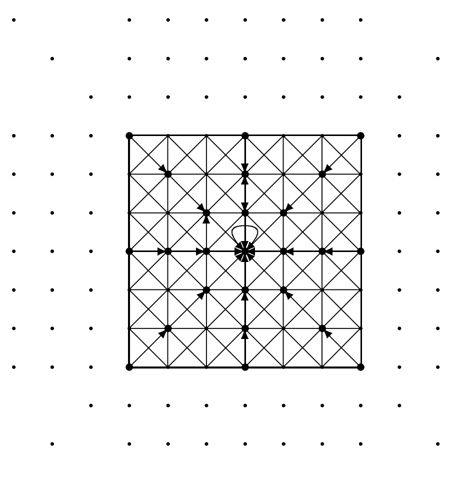


Figure 11

By the mentioned definition we can consider a new class of km-graphs that we will call them M_n*-graph maps*. Here the n is the union of singular vertexes with conclusive point. We can think on km-graphs by a loop in the conclusive point. The idea of such km-graphs come back to a square in city, when we consider the traffic problem, or the river in a sea/ocean, when we consider all rivers that have a sea/ocean as conclusive point. There fore we have the following definition:

Definition 4.9 *We will call a km-graph looped graph map if the conclusive point have a loop.*

Definition 4.10 *The sum of two km-graph G_{F_1} and G_{F_2} is a km-graph G_F that just obtained by superposition of conclusive points of G_{F_1} and G_{F_2} such that there is no sequence of vertexes such that*

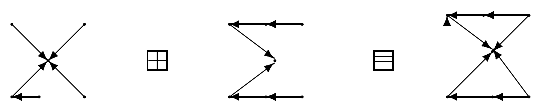
$$v_1 \longrightarrow v_2 \longrightarrow v_3 \longrightarrow \cdots \longrightarrow v_n \longrightarrow v_1,$$

or there is no close path such that

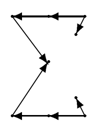
$$v_1 \longrightarrow v_2 \longrightarrow v_3 \longrightarrow \cdots \longrightarrow v_n = v_1 \longrightarrow v_{2'} \longrightarrow v_{3'} \longrightarrow \cdots \longrightarrow v_n,$$

Otherwise, by deleting one edge that have smallest number of input vertex we can obtain a km-graph.

One can see that for every two km-graph G_{F_1} and G_{F_2}, the difference between sum of G_{F_1} with G_{F_2} and G_{F_2} with G_{F_1} is only in the place of M. Such that in sum processing, we do it by superposition of conclusive point of second km-graph on conclusive point of first km-graph.



(a)



(b)

Figure 12

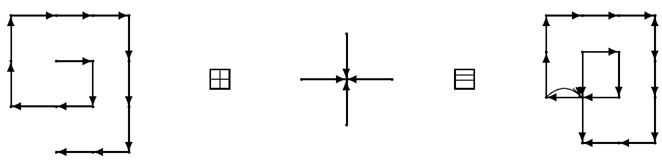
In the following one can find sum of two km-graph with restriction law:

Also if in sum of two km-graph we have two direction that are inverse, then we can use from this fact that, by definition, we have not closed path, the result form (a) can be considered as picture (b). That is we delete such paths. This law was called restricted law.

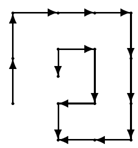
On the other hand we will see that the set of km-graphs with binary operation $\boxplus$ is semi group. Now if we consider the km-graph M, that is the km-graph with only one node and whit out vertex, the identity element of the set of all km-graphs, then we have a monoid that defined on set of all km-graphs with the binary operation $\boxplus$ because we can see that

$$(G_F 1 \boxplus G_F 2) \boxplus G_F 3 = G_F 1 \boxplus (G_F 2 \boxplus G_F 3)$$

$$G_F \boxplus M = M \boxplus G_F = G_F$$

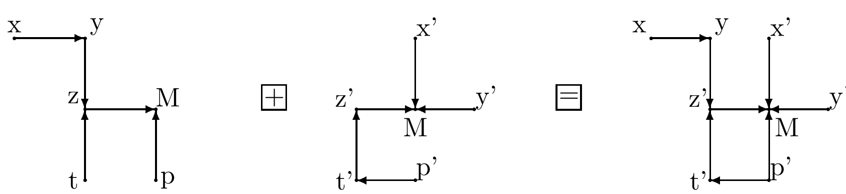


(a)



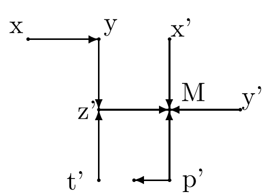
(b)

Figure 13



(1) (2) (3)

(a)



(b)

Figure 14 Samples for sum of two km-graphs

Also one can defined sum of looped km-graphs analogically.

Definition 4.11 *The sum of two looped km-graph G_{F_1} and G_{F_2} is a looped km-graph G_F that just obtained by superposition of conclusive points of G_{F_1} and G_{F_2} such that there is no sequence of vertexes such that*

$$v_1 \longrightarrow v_2 \longrightarrow v_3 \longrightarrow ... \longrightarrow v_n \longrightarrow v_1,$$

or there is no close path such that

$$v_1 \longrightarrow v_2 \longrightarrow v_3 \longrightarrow \ldots \longrightarrow v_n = v_1 \longrightarrow v_{2'} \longrightarrow v_{3'} \longrightarrow \ldots \longrightarrow v_n,$$

Otherwise, by deleting one edge that have smallest number of input vertex we can obtain a km-graph.

In the following we consider some samples of sum of two looped km-graphs.It is obvious that we can do this definition for km-graph from one hand and looped km-graph from other hand.

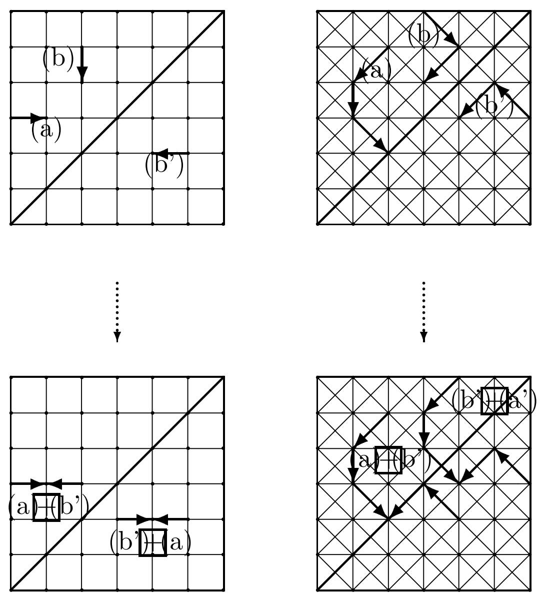
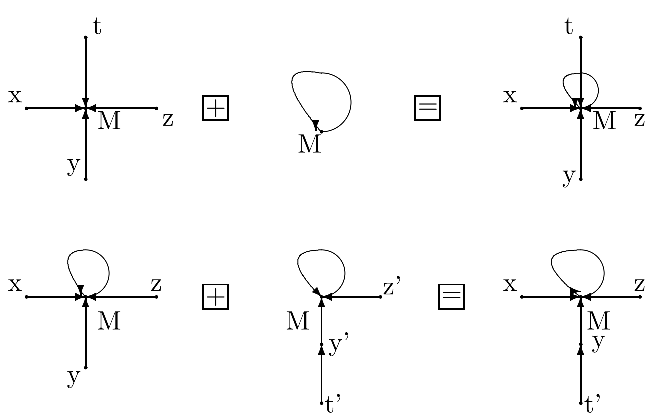


Figure 16 Samples for sum of two km-graphs



Sum of two looped graph function

Figure 17 Samples for sum of two graph function

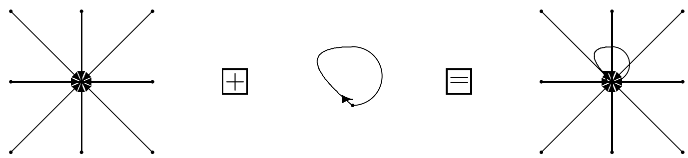
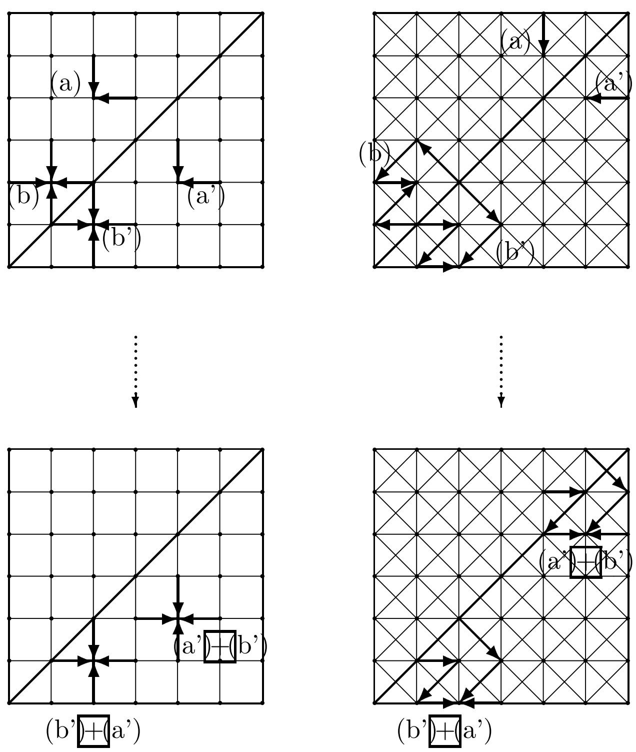


Figure 18 Samples for sum of two graph function

We can define $-G_F$ as a km-graph that is in negative part of graph plane. There fore we have $G_{F_1} \boxminus G_{F_2} = G_{F_1} \boxplus (-G_{F_2})$. On the other hand for $-G_{F_1} - GF_2$ we can consider $-(G_{F_1} \boxplus (-G_{F_2}))$ that is equal with $-(GF_1 \boxplus G_{F_2})$. In the following we have some samples for combining of such km-graphs:



Sum of two negative graph function

Figure 19 Samples for sum of two km-graphs

Therefore if we observe positive scalar multiplication and the negative part of an km-graph, then we can define negative scalar multiplication of a km-graph analogously. In this manner we can first multiple any scalar k in km-graph G_F and then found the negative position of this map. On the other hand we can first found for every km-graph G_F the negative position of G_F and then multiple $(-G_F)$ in k, for every arbitrary scalar $k \in \mathbb{Z}$.

4.1 Algebraic Properties of km-Graphs

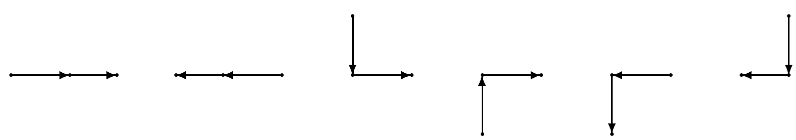
First of all we will consider associativity and commutativity properties for sum of km-graphs. Sum of two km-graphs are defined in previous section. The drawing km-graph for sum of G_{F_1} with G_{F_2} and G_{F_2} with G_{F_1} are similar, but are in different place of graph plane. There fore one can conclude that the sum of two km-graph is approx-commutative. Also one can see that the approx-associativity property hold for sum of three km-graphs.

On the other hand sum of every km-graph G_{F_1} with km-graph G_{F_M}, is G_{F_1} (where G_{F_M} is the km-graph with only one vertex and no edges). Thus, one can conclude that the class of all km-graphs considerable as a approx-commutative monoid (the approx-commutative semi-group with identity element).

On the other hand if we consider the km-graph as objects and the binary operation between them be the sum of two km-graphs, then can we have a category?

In this direction one can discussed that which properties of categories are satisfies in mentioned category and vise verse. It is obvious that by definition of km-graph and drawing them we have the following lemmas:

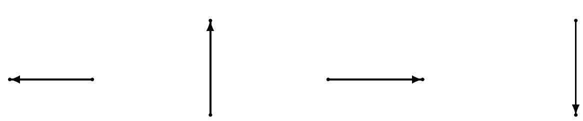
Lemma 4.1 *There are only one isomorphic km-graph with two nods in the $p_n * p_n$ plane.*



(a)

(b)

The classes of 3-vertex graph function



The classes of 2-vertex graph function

Figure 19

Lemma 4.2 *There are only two isomorphic class of km-graphs with three nods in the $p_n * p_n$ plane.*

Lemma 4.3 *There are only four class of km-graphs with four nods in the $p_n * p_n$ plane.*

4.2 km-Algebras

Definition 4.12 *We call an algebra defined on km-graphs a* $\mathbf{KM-Algebra}$*, if the following laws holds:*

$$xy \approx y \tag{1}$$

$$yx \approx \infty \tag{2}$$

$$MM \approx \infty \tag{3}$$

$$yM \approx \infty \tag{4}$$

$$My \approx \infty \tag{5}$$

$$M\infty \approx \infty \tag{6}$$

$$\infty M \approx \infty \tag{7}$$

$$y\infty \approx \infty \tag{8}$$

$$\infty y \approx \infty \tag{9}$$

$$y(yM) \approx yM \tag{10}$$

$$M(yM) \approx MM \approx \infty \tag{11}$$

$$(xy)z \approx (xy)(yz) \approx yz \tag{12}$$

$$x_1x_2x_3\cdots x_ny \approx y_1y_2y_3\cdots y_my \tag{13}$$

Equational theory on km-algebras can be discussed for identities and hyperidentities in nontrivial terms.

In [1], Baker, McNulty, and Werner give a method, the Shift-Automorphism Theorem, for showing that certain algebras are INFB. This method is particularly useful in the case of graph algebras; it is an essential ingredient in the classification in [1] of the FB graph algebras. The shift-automorphism theorem 4.5 can be used to show that many directed graph algebras are INFB as well, and we shall use it to obtain several such results in this chapter.

The form of the shift-automorphism theorem that we shall use is Theorem in [1], and it appears below as our . In an algebra that has an absorbing element ∞, as do all directed graph algebras, the proper elements are the elements other than ∞. Given a $\mathbf{Z}-$sequence α, the translates of α are the sequences α^i for $i \in \mathbf{Z}$, where α^i is α shifted i places to the right. (Thus if $i < 0$, then we shift to the left, and if $i = 0$ then we do not shift at all.)

Theorem 4.1(Shift-Automorphism Theorem) *Let B be a finite algebra of finite type, with an absorbing element ∞. Suppose that a sequence α of proper elements of B can be found with these properties:*

(1) *in $B^{\mathbf{z}}$, any fundamental operation f applied to translates of α yields as a value either a translate of α or a sequence containing ∞;*

(2) *there are only finitely many equations* $f(\alpha^{i_1}, \cdots, \alpha^{i_n(f)}) = \alpha^{(j)}$ *in which* f *is a fundamental operation and some argument is a* α *itself;*

(3) *there is at least one equation* $f(\alpha^{i_1}, \cdots, \alpha^{i_n(f)}) = \alpha^{(1)}$ *in which some argument is a itself, in an entry on which* f *actually depends,*

then B *is INFB.*

Proof See [23]. □

The general idea of this version of the Shift-Automorphism Theorem, as applied specifically to a directed graph algebra B, is that we want to use α to create an infinite directed graph with certain properties. The elements of this directed graph are the translates of α, and the edges are the natural ones inherited from G_B. We pool all sequences that contain ∞ into an equivalence class, which acts as the ∞ element for resulting infinite directed graph algebra. Condition (a) of the theorem ensures that this view makes sense. Condition (b) requires there to be only finitely many edges into and out of α (and therefore into and out of any $\alpha^{(i)}$); another way to say this is that there must be an N such that if $n \geq N$, then $\alpha\alpha^{(n)}$ and $\alpha^{(n)}\alpha$ must both contain an occurrence of ∞. (Here $\alpha^{(i)}\alpha^{(j)}$ is understood to be the result of applying our binary operation coordinate wise to $\alpha^{(i)}$ and $\alpha^{(j)}$.) Condition (c) tells us that there has to be edge from $\alpha^{(1)}$ to α.

Because the multiplication in our km-algebras has the property that uv is either v or ∞, if B is a KM-Algebra and α is any sequence of proper elements of B, then it is clear that condition (a) of the theorem is satisfied; in each coordinate of α, the product will either be the right-hand operand or ∞. Hence we do not need to mention condition (a) again when dealing with km-algebra. Note also that if α is an infinite path through G_B, as it will be in all of the cases we consider, then condition (c) must hold.

The original version of the shift automorphism theorem, as formulated in 1989 by Baker, McNulty, and Werner [1], stated that any shift automorphism algebra is inherently nonfinitely based. In 2008, McNulty, Szekly, and Willard were able to show every shift automorphism algebra must be inherently nonfinitely based in the finite sense [16]. It is the contribution of this author that every shift automorphism variety has countably infinite sub directly irreducible members.

For some interesting examples of finite algebra proven to be inherently nonfinitely based with help of the Shift Automorphism Theorem, see [1, 6, 10, 23]. As an example of the shift automorphism theorem, let us consider the looped star-like km-algebra; it is based on the km-graph pictured in figure(10). We let $\alpha = \cdots aaaaabcccc \cdots$. Now, α is an infinite path through looped star-like km-algebra, so we simply need to show that condition (b) of the Shift-Automorphism Theorem holds for this algebra and α.

To begin, let us observe that if $i \notin \{j, j+1\}$, then $\alpha^{(i)}\alpha^{(j)}$ will be a sequence that contains ∞, since there will be at least one coordinate at which the entry is ac, ba, cb, or ca, and all of these are ∞. If $i \in \{j, j+1\}$, then $\alpha^{(i)}\alpha^{(j)}$, then $\alpha^{(i)}\alpha^{(j)} = \alpha^{(i)}$. Thus α meets condition (b) of the theorem; the only equations that hold here of the kind mentioned in (b) are $\alpha^{(0)}\alpha^{(0)} = \alpha^{(0)}$, $\alpha^{(0)}\alpha^{(-1)} = \alpha^{(0)}$ and $\alpha^{(1)}\alpha^{(0)} = \alpha^{(1)}$.

Thus we have proven the following:

Lemma 4.4 *For $n \geq 3$ every n-Starlike km-algebra is INFB.*

Also for the looped n-starlike km-algebra the above lemma is true. The infinite looped star-like km-algebra that α yields is pictured in Figure 10.

Note that the same α would have worked even if the km-algebra in question had either (or both) of the edges (b, a) and (c, b). Thus the km-algebras in Figure 10 give rise to INFB km-algebras as well.

Also the nullary looped km-algebra (with one vertex and one edges) is just a single looped element, so is FB. Thus we have now completely classified the looped star-like km-algebras.

To use the Shift-Automorphism Method, consider **B**, a finite algebra of finite signature. We will consider the $\mathbb{Z}-$tuple $(\cdots, b_{-2}, b_{-1}, b_0, b_1, b_2, \cdots)$ in $\mathbf{B}^{\mathbb{Z}}$ as a sequence $\alpha = \cdots b_{-2}b_{-1}b_0b_1b_2\cdots$ where each b_j is a proper element of **B** for every $j \in \mathbb{Z}$. We define α_i as the i^{th} translate of α, that is, as α shifted $i$positions to the right (if $i > 0$), to the left (if $i < 0$), or not at all (if $i = 0$). Note that the shift by 1 position is an automorphism of $\mathbf{B}^{\mathbb{Z}}$. If σ gives only one infinite orbit, then we can summarize Theorem 4.1 as the following, also found on [1].

Theorem 4.2 *Let* **B** *be a finite algebra of finite signature with absorbing element* 0. *Suppose that a sequence α of proper elements of* **B** *can be found with these properties:*

(1) *in $\mathbf{B}^{\mathbb{Z}}$, any fundamental operation F applied translates of α yields as a value either a translate of α or a sequence containing 0;*

(2) *there are only finitely many equations $(\alpha_{i_0}, \cdots, \alpha_{i_{r-1}}) = \alpha_j$ in which F is a fundamental operation of rank r and some argument is α itself;*

(3) *there is at least one equation $F(\alpha_{i_0}, \cdots, \alpha_{i_{r-1}}) = \alpha_1$ in which some argument is α itself, in an entry on which F actually depends.*

We apply Theorems 4.1 and 4.2 to various algebras to show that they are inherently nonfinitely based.

4.3 Walter's Looped Directed Graphs

Walter gives the following adapted version of Theorem 4.2.

Theorem 4.3([23]) *Let G be a directed graph, and let α be a $\mathbb{Z}-$sequence that is a path of* **G**. *If there is an N such that $n > N$ implies that $\alpha_n.\alpha$ and $\alpha.\alpha_n$ both contain an occurrence of ∞, then the graph algebra of* **G** *in inherently nonfinitely based.*

Now for our km-algebras we have another version of Theorem 4.2.

Theorem 4.4 *Let* **G** *be a KM- algebra, and α be a $\mathbb{Z}-$sequence that is a path of* **G**. *If there is an N such that $n > N$ implies that $\alpha_n.\alpha$ and $\alpha.\alpha_n$ both contain an occurrence of ∞, then the km-algebra G is inherently nonfinitely based.*

Proof We will show the connection between Theorems 4.1 and 4.3. Let μ be a km-algebra and let α be a $\mathbb{Z}$-sequence that is a path through **B**. We will check the conditions of Theorem 4.1.

The first condition of Theorem 4.1 requires that any fundamental operation applied to any translate of α results in another translate of α, or a sequence containing ∞. As our operation is inherited from a graph algebra, the operation. works as follows:

$$u.v = \begin{cases} v & if(u,v) \in E \\ \infty & otherwise \end{cases}$$

Thus for the coordinate wise product of two $Z-$sequence α_j, α_k we get

$$\alpha_j, \alpha_k \begin{cases} \alpha_k & \\ \gamma & where \gamma is a sequence containing \infty \end{cases}$$

since the result of applying. Coordinate wise gives either right input or ∞.

The second condition of Theorem 4.1 requires that there be only finitely many proper equations using the fundamental operation and the translates of α. In terms of our new infinite km-algebra, this conditions there to only finitely many edge into and out of each α_i. This is equivalent to having the existence of a number N so that for all $n > N$ we have $\alpha_n.\alpha$ and $\alpha.\alpha_n$ each contain an occurrence of ∞.

Since α_1 is a path through the km-algebra, $\alpha_1.\alpha_0 = \alpha_0$. To see this, consider a string $\cdots x_{-3}x_{-2}x_{-1}x_0x_1x_2x_3\cdots$ in α_1 where each x_i is a vertex in the km-graph related to km-algebra. Then α_1 has the same string, just shifted one place rightward. Since α_1 gives a path, we know that must be an edge between x_i and x_{i+1} for $i = -1, ...4$. Thus the product $x_i.x_{i+1}$ results x_{i+1}. Hence we get the following

$$\alpha_1 : \cdots x_{-1}x_0x_1x_2x_3\cdots$$

$$\alpha_0 : \cdots x_0x_1x_2x_3x_4\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\alpha_0 : \cdots x_0x_1x_2x_3x_4\cdots$$

This gives the last condition of Theorem 4.1. □

§5. Basic Laws for Directed M_n-km-Graphs

Definition 5.1

$$M_1M_1 \approx \infty \tag{14}$$

$$M_1x \approx \infty \tag{15}$$

$$xM_1 \approx M_1 \tag{16}$$

$$x\infty \approx \infty \tag{17}$$

$$\infty x \approx \infty \tag{18}$$

$$xx \approx \infty \tag{19}$$

$$xy \approx y \tag{20}$$

$$(xM_1)M_1 \approx M_1M_1 \approx \infty \tag{21}$$

$$x_1x_2 \cdots x_nM_1 \approx y_1y_2 \cdots y_mM_1 \tag{22}$$

For discrete points, $M_iM_j \approx \infty$.

Lemma 5.1 *The identity* $(xM_1)M_1 \approx M_1(xM_1)$ *holds.*

Proof $(xM_1)M_1 \approx$ by (16), $(xM_1)(xM_1) \approx$ by (16) and then $\approx M_1(xM_1)$. □

§6. Basic Laws for Directed Looped M_n-km-Graphs

Definition 6.1

$$M_1M_1 \approx M_1 \tag{23}$$

$$M_1x \approx \infty \tag{24}$$

$$xM_1 \approx M_1 \tag{25}$$

$$x\infty \approx \infty \tag{26}$$

$$\infty x \approx \infty \tag{27}$$

$$xx \approx \infty \tag{29}$$

$$xy \approx y \tag{30}$$

$$(xy)y \approx yy \approx \infty \tag{31}$$

$$x_1x_2 \cdots x_nM \approx y_1y_2 \cdots y_mM \tag{32}$$

For discrete points, $M_iM_j \approx \infty$.

Lemma 6.1 *The identity* $(xy)y \approx y(xy)$ *holds.*

Proof $(xy)y \approx$ by (29), $yy \approx$ by (29) and then $\approx y(xy)$. □

§7. Basic Laws for Looped km-Algebra

Definition 7.1

$$MM \approx M \tag{33}$$

$$yM \approx M \tag{34}$$

$$My \approx \infty \tag{35}$$

$$M\infty \approx \infty \tag{36}$$

$$\infty M \approx \infty \tag{37}$$

$$y(yM) \approx yM \approx M \tag{38}$$

$$(xy)M \approx yM \approx M \tag{39}$$

$$(yM)M \approx MM \approx \infty \tag{40}$$

$$M(yM) \approx MM \approx \infty \tag{41}$$

$$x_1 x_2 \cdots x_n M \approx y_1 y_2 \cdots y_m M \tag{42}$$

Lemma 7.1 *The identity $(yM)M \approx M(yM)$ holds.*

Proof $(yM)M \approx$ by (34), $MM \approx$ by (34) and then $\approx M(yM)$. □

It turns out that the theory of avoidable words will be quite useful. We will begin with a bunch of definitions.

Definition 7.1 *An alphabet* $\mathbf{\Sigma}$ *is a set of letters and a letter is a member of some alphabet. A word is a finite string of letters from some alphabet. The* **empty word** *is the word of length* 0. *To denote the set of all nonempty words over an alphabet* Σ*, we use* Σ^+*. Another formulation of* Σ^+ *is that it is the semi group freely generated by* Σ *together with the binary operation of concatenation.*

Definition 7.2 *A word w is an instance of a word u provided that w can be obtained from u by substituting nonempty words for the letters of u.*

For instance, the word baabaabaabaa is an instance of the word $xxxx$ obtained by substituting the word baa for the letter x. We say that a word u is a sub word of the word w if there are (potentially) words x and y so that $w = xuy$.

Definition 7.3 *The word w encounters the word u means that some instance of u is a sub word of w. If no instance of u is a sub word of w, then we say w avoids u.*

To generalize this notion, we say that the word u is *avoidable the alphabet* Σ provided that infinitely many words in Σ^+ avoid u. Note that since two alphabets of the same size avoid the same words, only the cardinality of Σ is important. If the cardinality of Σ is n and u is avoidable on Σ, then we say that u is *n-avoidable.* Lastly, the word u is *avoidable* if and only if there is some natural number n for which u is *n-avoidable.* If no such n exists, we call u *unavoidable.*

Definition 7.4 *The Zimin words Z_n (where n is a natural number) are defined recursively by:*

(i) $Z_0 = x_0$;

(ii) Given Z_n*, define* $Z_{n+1} = Z_n x_{n+1} Z_n$ *for each natural number* n.

Thus the first three Zimin words are $x_0, x_0x_1x_0$, and $x_0x_1x_0x_2x_0x_1x_0$. If S is a semi group and w is a word in which the letters of w are regarded as variables, we say w is an *isoterm of* S when u and w are identical whenever $\mathbf{S} \models w \approx u$. See [26] for a development of the theory of avoidable words.

Definition 7.5 *An algebra* $\mathbf{B} =< B, . >$ *is a* **semigroup** *provided the associative law holds, i.e. for all* $a, b, c \in B$, $(a.b).c = a.(b.c)$.

Parkins in [19] constructs the following semigroup, denoted by $\mathbf{B_2^1}$:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

We denoted these matrices by $\mathbf{O}$, $\mathbf{I}$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$, respectively. The semi group structure of this algebra is given by Table 3.

Perkins's semi group is inherently nonfinitely based, as shown by Sapir in [7, 8]. To show this, we need a theorem of Sapir from [7, 8].

Theorem 7.1 *Let* $\mathbf{S}$ *be a finite semi group. If every unavoidable word is an isoterm of* $\mathbf{S}$*, then* $\mathbf{S}$ *is inherently nonfinitely based.*

.	O	I	A	B	C	D
O	O	O	O	O	O	O
I	O	I	A	B	C	D
A	O	A	A	B	O	O
B	O	B	O	O	A	B
C	O	C	C	D	O	O
D	O	D	O	O	C	D

TABLE (*) of the Semi group table of $\mathbf{B_2^1}$

We note that Sapir also showed the more difficult converse, that if $\mathbf{S}$ is inherently nonfinitely based then every unavoidable word is an isoterm of $\mathbf{S}$. For its proof, see [9].

Corollary 7.1 *The semigroup* $\mathbf{B_2^1}$ *is inherently nonfinitely based.*

Proof See [9]. □

Theorem 7.2 *The set of six km-algebra(shown below) with binary Operation, is isomorphic by Perkin's semi group. Therefore we have inherently nonfinitely based algebra for mentioned six km-algebra.*

Proof The result obtained from Theorem 4.1 immediately. Correspondence of these six km-algebra and Parkin's semi group shown in the following figure. □

By similar method we will see that many of known $INFB$ algebras are isomorphic with some subsets of M and parkin's semigroup shown in the following. Therefore,

Theorem 7.3 *The variety of all km-algebras is inherently nonfinitely based.*

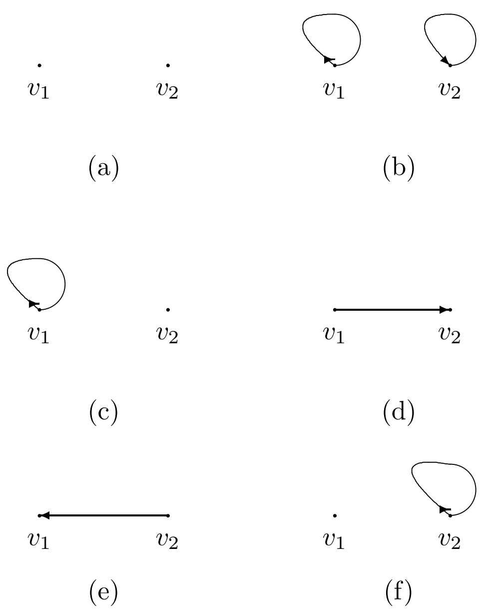


Figure 20

O	v_1	v_2
v_1	0	0
v_2	0	0

Table a

v_1	v_2	
v_1	1	0
v_2	0	1

Table b

O	v_1	v_2
v_1	1	0
v_2	0	0

Table c

O	v_1	v_2
v_1	0	1
v_2	0	0

Table d

O	v_1	v_2
v_1	0	0
v_2	1	0

Table e

O	v_1	v_2
v_1	0	0
v_2	0	1

Table f

We will see that (by definition of operation $\boxplus$) sum of two M_n km-graph can be considered as one of the following pictures:

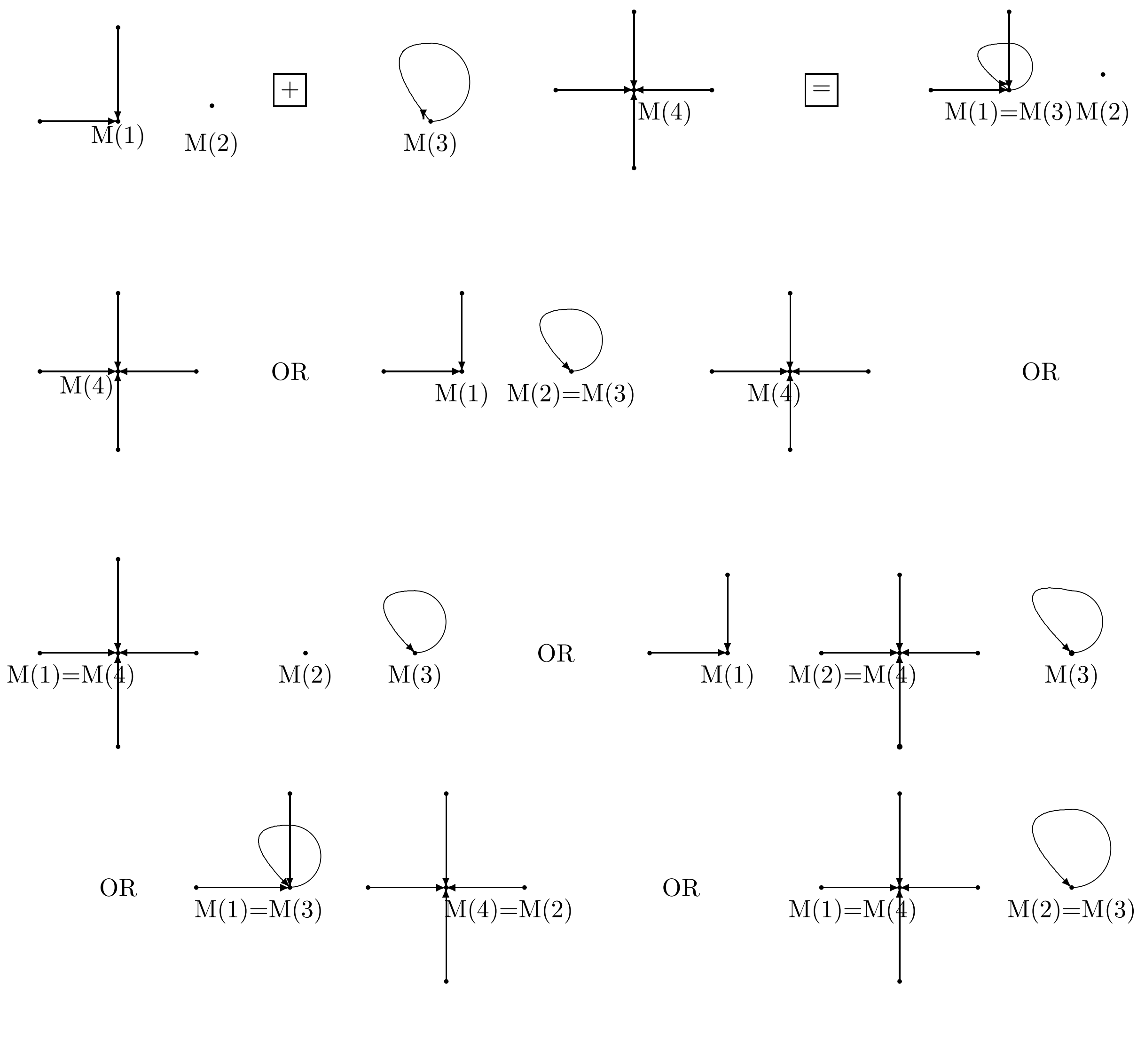


Figure 21 Sum of two $M_n$km-graphs

References

[1] Kirby A. Baker, G. F. McNulty, H. Werner, The finitely based varieties of graph algebras, *Acta Scient. Math.*, 51: 3-15, (1987).

[2] Garrett Birkhoff, On Structure of Abstract Algebras, *Proc. Cambridge Philos. Soc.*, 31 (1935), 433-454.

[3] Wieslaw Dziobiak, On infinite vsubdirectly irreducible algebras in locally finite equational classes, *Algebra Universelis*, 13 (1981), No. 3, 393-394.

[4] Samuel Eilenberg, M. P. Schutzenberger, On pseudovarieties, *Advances in Math.*, 19 (1976), No. 3, 413-418.

[5] E. Hajilarov, An inherently nonfinitely based commutative directoid, *Algebra Universelis*, 36 (1996), No.4, 431-435.

[6] Keith Kearnes and Ross Willard, Inherently nonfinitely based solvable algebras, *Canad. Math. Bull.*, 37 (1994), No.4, 514-521.

[7] M. V. Sapir, Inherently nonfinitely based finite semigroups, *Mat. Sb. (N.S.)*, 133 (175) (1987), no. 2, 154-166.

[8] M. V. Sapir, Problems of Burnside type and the finite basis property in varieties of semigroups, *Izv. Akad. Nauk SSSR Ser. Mat.*, 51 (1987), no. 2, 319-340.

[9] Kathryn Hope Scott, *A Catalogue of Finite Algebras with Nonfinitely Axiomatizable Equational theories*, Ph.D. Thesis, University of south Carolina, 2007.

[10] Agnes Szendrei, Non-finitely based finite groupoids generating minimal varieties, *Acta Sci. Math.* (Szeged), 57 (1993), No. 1-4, 593-600.

[11] W.Kiss, R. Poschel, P.prohle, Subvarieties of varieties generated by graph algebras, *Acta Scient. Math.*,54(1-2):57-57,(1990).

[12] Ralph McKenzie, George F. McNulty, Walter F. Taylor, *Algebras, Lattices, Varieties.* Vol I, The Wads worth and Brooks/cole Mathematics Series, Wads worth and Brooks/cole Advanced Books and Software, Monterey, CA, 1987.

[13] Ralph McKenzie,The residual bounds of finite algebras, *Internat. J. Algebra Comput.*, 6 (1996), No.1, 1-28.

[14] Ralph McKenzie, Tarski's finite basis problem is undecidable, *Internat. J. Algebra Comput.*, 6 (1996), No. 1, 49-104.

[15] Georg F. McNulty, Residual finiteness and finite equational bases: undecidable properties of finite algebras, *Lectures on Some Recent Works of Ralph McKenzie and Ross Willard.*

[16] Georg F. McNulty, Zoltan Szekely, Ross Willard, Equational complexity of the finite algebra membership problem, *IJAC*, 18 (2008), 1283-1319.

[17] J. Jezek, G. F. McNulty, Finite axiomatizablity of congruence rich varieties, *Algebra Universelis*, 34 (1995), no.2, 191-213.

[18] A.Kaveh, M.Nouri, Weighted graph products for configuration processing of planer and space structures, *International Journal of Space Structure*, Vol.24, Number 1,(2009).

[19] Robert E. Park, *Equational classes on non-associative ordered algebras*, Ph.D. thesis, University of California, Los Angeles, 1976.

[20] Robert W. Quackenbush, Equational classes generated by finite algebras, *Algebra Universelis*, 1 (1971/72), 265-266.

[21] Kathryn Scott Owens, *On Inherently Nonfinitely Based Varieties.* Ph.D. Thesis, University of south Carolina, 2009.

[22] L. Wald, *Minimal inherently nonfinitely based varieties of groupoids*, Ph.D. thesis, University of California, Los Angeles, 1998.

[23] Brin L.Walter, Finite Equntional Bases for Directed Graph Algebras. Ph.D. Thesis,University of California, 2002.

[24] Brian L.Walter, The finitely based varieties of looped directed graph algebras, *Acta Sci. Math* (Szeged), 72(2006), No.3-4, 421-458.

[25] Emil W. Kiss, R. Poschel, P. Prohle, Subvarieties of varieties generated by graph algebras, *Acta Scient. Math.*, 54(1-2): 57-75, (1990).

[26] Ross Willard, The finite basis problem, *Contributions to General Algebra*, 15, Heyn, Klagenfurt, 2004, pp. 199-206.

Math.Combin.Book Ser. Vol.3(2015), 33-42

Superior Edge Bimagic Labelling

R.Jagadesh

(Department of Mathematics, Easwari Engineering College, Chennai - 6000 089, India)

J.Baskar Babujee

(Department of Mathematics, MIT Campus, Anna University, Chennai - 600 044, India)

E-mail: jagadesh_rajagopal@yahoo.com, baskarbabujee@yahoo.com

Abstract: A graph $G(p, q)$ is said to be edge bimagic total labeling with two common edge counts k_1 and k_2 if there exists a bijection $f : V \cup E \to \{1, 2, \cdots, p+q\}$ such that for each edge $uv \in E$, $f(u) + f(v) + f(e) = k_1$ or k_2. A total edge bimagic graph is called superior edge bimagic if $f(E(G)) = \{1, 2, \cdots, q\}$. In this paper we have proved superior edge bimagic labeling for certain class of graphs arising from graph operations.

Key Words: Graph, magic labeling, bijective function, edge bimagic, superior edge bimagic labeling.

AMS(2010): 05C78.

§1. Introduction

A labelling of a graph G is an assignment f of labels to either the vertices or the edges or both subject to certain conditions. Labeled graphs are becoming an increasingly useful family of mathematical Models from broad range of applications.Graph labelling was first introduced in the late 1960's. A useful survey on graph labelling by J.A. Gallian (2013) can be found in [1]. All the graphs considered here are finite, simple and undirected. We follow the notation and terminology of [2]. In most applications labels are positive (or nonnegative) integers, though in general real numbers could be used.

A (p, q)-graph $G = (V, E)$ with p vertices and q edges is called total edge magic if there is a bijection $f : V \cup E \to \{1, 2, \cdots, p+q\}$ such that there exists a constant k for any edge uv in E, $f(u) + f(uv) + f(v) = k$. The original concept of total edge-magic graph is due to Kotzig and Rosa [3]. They called it magic graph. A total edge-magic graph is called a superior edge-magic if $f(E(G)) = \{1, 2, \cdots, q\}$.

It becomes interesting when we arrive with magic type labeling summing to exactly two distinct constants say k_1 or k_2. Edge bimagic total labeling was introduced by J. Baskar Babujee [6]and studied in [7] as (1, 1) edge bimagic labeling. A graph $G(p, q)$ with p vertices and q edges is called total edge bimagic if there exists a bijection $f : V \cup E \to \{1, 2, \cdots, p+q\}$ such that

[1]Received December 5, 2014, Accepted August 7, 2015.

for any edge $uv \in E$, we have two constants k_1 and k_2 with $f(u)+f(v)+f(uv)=k_1$ or k_2. A total edge-bimagic graph is called superior edge bimagic if $f(E(G))=\{1,2,\cdots,q\}$. Superior edge bimagic labelling was introduced and studied in [8].

Definition 1.1 *A pyramid graph $PY(n)$ is obtained from Prism graph $P_n \times C_3$ whose $V(P_n \times C_3)=\{v_{ij} : 1 \leq i \leq n, 1 \leq j \leq 3\}$ by adding a new vertex v_{00} adjacent to the three vertices v_{11}, v_{12}, v_{13} of $P_n \times C_3$. This graph has $3n+1$ vertices and $6n$ edges.*

Definition 1.2 *mK_n - Snake is a connected graph with m blocks whose block-cut point graph is a path and each of the m blocks is isomorphic to Complete graph K_n.*

Definition 1.3 *mW_n - Snake is a connected graph with m blocks whose block-cut point graph is a path and each of the m blocks is isomorphic to Wheel graph W_n.*

Definition 1.4 *A graph $G(p,q)$ is said to have an edge magic total labeling with common edge counts k_0 if there exists a bijection $f : V \cup E \to \{1,2,\cdots,p+q\}$ such that for each $e=(u,v) \in E$, $f(u)+f(v)+f(e)=k_0$. A total edge magic graph is called superior edge-magic if $f(E(G))=\{1,2,\cdots,q\}$.*

Definition 1.5 *A graph $G(p,q)$ is said to be edge bimagic total labeling with two common edge count k_1 and k_2 if there exists a bijection $f : V \cup E \to \{1,2,\cdots,p+q\}$ such that for each $e=(u,v) \in E$, $f(u)+f(v)+f(e)=k_1$ or k_2. A total edge-bimagic graph is called superior edge-bimagic if $f(E(G))=\{1,2,\cdots,q\}$.*

Definition 1.6 *If $G_1(p_1,q_1)$ and $G_2(p_2,q_2)$ are two connected graphs, $G_1\hat{O}G_2$ is obtained by superimposing any selected vertex of G_2 on any selected vertex of G_1. The resultant graph G belongs to the class $G_1\hat{O}G_2$ consists of p_1+p_2-1 vertices and q_1+q_2 edges. In general, we can construct p_1p_2 possible combination of graphs from G_1 and G_2.*

§2. Superior Edge Bimagic Labeling for Special Class of Graphs

Theorem 2.1 *A pyramid graph $PY(n)$ is superior edge bimagic for $n \geq 3$.*

Proof Let $f : V \cup E \to \{1,2,3,\cdots,9n+1\}$ be a bijection defined by

(i) $f(v_{00})=6n+1$, $f(v_{00}v_{11})=6n$, $f(v_{00}v_{12})=6n-1$, $f(v_{00}v_{13})=6n-2$, and

(ii) $f(v_{3i-2,1})=9i+6n-7$, $f(v_{3i-2,2})=9i+6n-6$, $f(v_{3i-2,3})=9i+6n-5$, $f(v_{3j-1,1})=9j+6n-2$, $f(v_{3j-1,2})=9j+6n-4$, $f(v_{3j-1,3})=9j+6n-3$, $f(v_{3k,1})=9k+6n$, $f(v_{3k,2})=9k+6n+1$, $f(v_{3k,3})=9k+6n-1$, $f(v_{3i-2,1}v_{3i-2,2})=6n-18i+15$, $f(v_{3i-2,2}v_{3i-2,3})=6n-18i+13$, $f(v_{3i-2,3}v_{3i-2,1})=6n-18i+14$, $f(v_{3j-1,1}v_{3j-1,2})=6n-18j+8$, $f(v_{3j-1,2}v_{3j-1,3})=6n-18j+9$, $f(v_{3j-1,3}v_{3j-1,1})=6n-18j+7$, $f(v_{3k,1}v_{3k,2})=6n-18k+1$, $f(v_{3k,2}v_{3k,3})=6n-18k+2$, $f(v_{3k,3}v_{3k,1})=6n-18k+3$, where

(a) $i,j,k=1,2,3,\cdots,[n/3]$, when $n \equiv 0 \pmod 3$ and
(b) $i=1,2,3,\cdots,[n/3]+1$; $j,k=1,2,3,\cdots,[n/3]$, when $n \equiv 1 \pmod 3$,

(c) $i, j = 1, 2, 3, \cdots, [n/3] + 1$; $k = 1, 2, 3, \cdots, [n/3]$, when $n \equiv 2 \pmod 3$,

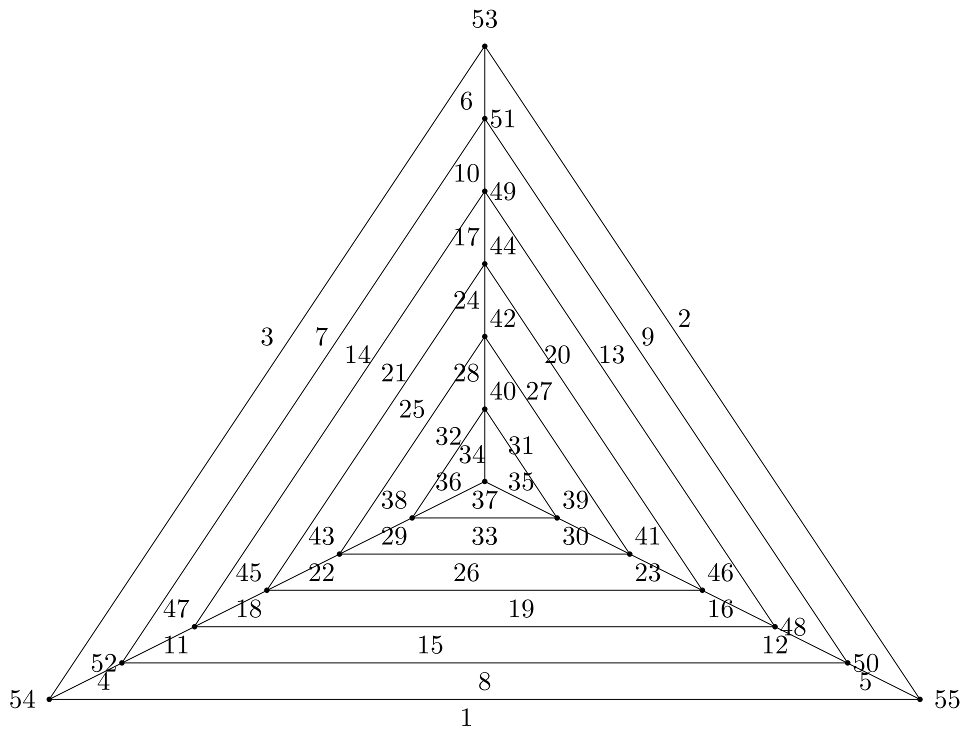


Figure 1 Superior bimagic of $PY(6)$ with $k_1 = 111, k_2 = 110$

(iii) $f(v_{3i-2,1}v_{3i-1,1}) = 6n-18i+11$, $f(v_{3i-2,2}v_{3i-1,2}) = 6n-18i+12$, $f(v_{3i-2,3}v_{3i-1,3}) = 6n-18i+10$, $f(v_{3j-1,1}v_{3j,1}) = 6n-18j+4$, $f(v_{3j-1,2}v_{3j,2}) = 6n-18j+5$, $f(v_{3j-1,3}v_{3j,3}) = 6n-18j+6$, $f(v_{3k,1}v_{3(k+1)-2,1}) = 6n-18k$, $f(v_{3k,2}v_{3(k+1)-2,2}) = 6n-18k-2$, $f(v_{3k,3}v_{3(k+1)-2,3}) = 6n-18k-1$, where

(a) $i, j = 1, 2, 3, \cdots, [n/3]$; $k = 1, 2, 3, \cdots, [n/3] - 1$ when $n \equiv 0 \pmod 3$,

(b) $i, j, k = 1, 2, 3, \cdots, [n/3]$, when $n \equiv 1 \pmod 3$,

(c) $i = 1, 2, 3, \cdots, [n/3] + 1$; $j, k = 1, 2, 3, \cdots [n/3]$, when $n \equiv 2 \pmod 3$.

We prove this labelling is superior edge bimagic. Now

$$f(v_{00}) + f(v_{11}) + f(v_{00}v_{11}) = 6n + 1 + 6n + 2 + 6n = 18n + 3,$$
$$f(v_{00}) + f(v_{12}) + f(v_{00}v_{12}) = 6n + 1 + 6n + 3 + 6n - 1 = 18n + 3,$$
$$f(v_{00}) + f(v_{13}) + f(v_{00}v_{13}) = 6n + 1 + 6n + 4 + 6n - 2 = 18n + 3.$$

Given n, considering appropriate values for i, j and k, we have

$$f(v_{3i-2,1})+f(v_{3i-2,2})+f(v_{3i-2,1}v_{3i-2,2}) = 9i+6n-7+9i+6n-6+6n-18i+15 = 18n+2,$$

$$f(v_{3i-2,2})+f(v_{3i-2,3})+f(v_{3i-2,2}v_{3i-2,3}) = 9i+6n-6+9i+6n-5+6n-18i+13 = 18n+2,$$

$$f(v_{3i-2,3})+f(v_{3i-2,1})+f(v_{3i-2,3}v_{3i-2,1}) = 9i+6n-5+9i+6n-7+6n-18i+14 = 18n+2,$$

$$f(v_{3j-1,1})+f(v_{3j-1,2})+f(v_{3j-1,1}v_{3j-1,2}) = 9j+6n-2+9j+6n-4+6n-18j+8 = 18n+2,$$

$$f(v_{3k,1}) + f(v_{3k,2}) + f(v_{3k,1}v_{3k,2}) = 9k + 6n + 9k + 6n + 1 + 6n - 18k + 1 = 18n + 2.$$

Also for the remaining any edge uv, the sums $f(u) + f(v) + f(uv) = 18n + 2$. Hence the graph $PY(n)$ admits superior edge bimagic labelling. □

The superior edge bimagic labelling of $PY(6)$ shown in the Figure 1.

Theorem 2.2 *A nK_4-Snake graph admits superior edge bimagic labelling for $n \geq 1$.*

Proof The vertex set of nK_4-Snake graph is given by $V = \{x_i : 1 \leq i \leq n+1\} \cup \{y_i, w_i : 1 \leq i \leq n\}$ and edge set is given by $E = \{x_ix_{i+1}, x_iy_i, y_ix_{i+1}, x_iw_i, w_ix_{i+1} : 1 \leq i \leq n\}$.

Consider the k^{th} block of nK_4-Snake graph label the vertices and edges as follows

$$f(x_k) = 3k + 6n - 2,\ f(x_{k+1}) = 3k + 6n + 1,\ f(y_k) = 3k + 6n,\ f(w_k) = 3k + 6n - 1,$$

$$f(x_kx_{k+1}) = 6n - 6k + 4,\ f(x_ky_k) = 6n - 6k + 5,\ f(y_kx_{k+1}) = 6n - 6k + 1,$$

$$f(x_kw_k) = 6n - 6k + 6,\ f(w_kx_{k+1}) = 6n - 6k + 2 \text{ and } f(w_ky_k) = 6n - 6k + 3.$$

A superior edge bimagic labelling of $4K_4$-Snake graph shown in the Figure 2.

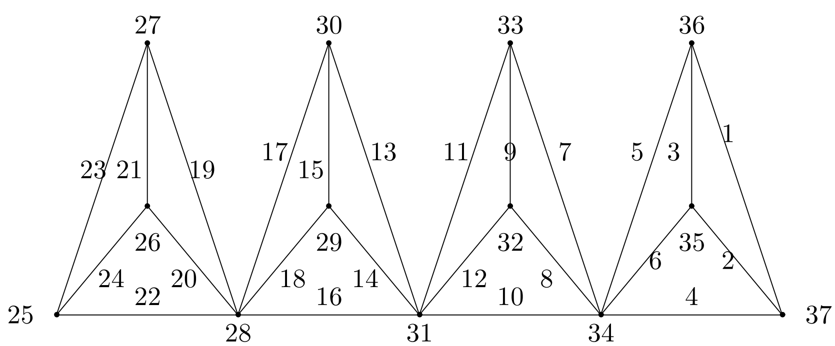


Figure 2 Superior edge bimagic of $4K_4$-Snake graph with $k_1 = 75$ and $k_2 = 74$

It is sufficient to prove that the k^{th} block of $4K_4$-Snake graph is superior edge bimagic where $1 \leq k \leq n$

$$f(x_k) + f(x_{k+1}) + f(x_kx_{k+1}) = 3k + 6n - 2 + 3k + 6n + 1 + 6n - 6k + 4 = 18n + 3,$$

$$f(x_k) + f(y_k) + f(x_ky_k) = 3k + 6n - 2 + 3k + 6n + 6n - 6k + 5 = 18n + 3,$$

$$f(y_k) + f(x_{k+1}) + f(y_kx_{k+1}) = 3k + 6n + 3k + 6n + 1 + 6n - 6k + 1 = 18n + 2,$$

$$f(x_k) + f(w_k) + f(x_kw_k) = 3k + 6n - 2 + 3k + 6n - 1 + 6n - 6k + 6 = 18n + 3,$$

$f(w_k)+f(x_{k+1})+f(w_kx_{k+1})=3k+6n-1+3k+6n+1+6n-6k+2=18n+2,$

$f(w_k)+f(y_k)+f(w_ky_k)=3k+6n-1+3k+6n+6n-6k+3=18n+2.$

Therefore for any edge uv, $f(u)+f(v)+f(uv)$ yields either $18n+3$ or $18n+2$. Hence the nK_4-Snake graph admits superior edge bimagic labelling. □

Theorem 2.3 *A nW_4-Snake graph admits superior edge bimagic labelling.*

Proof The vertex set of nW_4 is given by $V=\{x_i:1\le i\le n+1\}\cup\{y_i,z_i,w_i:1\le i\le n\}$ and edge set is given by $E=\{x_iy_i,x_iz_i,x_iw_i,z_ix_{i+1},y_ix_{i+1},w_ix_{i+1}:1\le i\le n\}$.

Consider the k^{th} block of nW_4 and label the vertices and edges as follows

$f(x_k)=4k+8n-3,\ f(y_k)=4k+8n-1,\ f(x_{k+1})=4k+8n+1,$

$f(z_k)=4k+8n-2,\ f(w_k)=4k+8n,\ f(x_ky_k)=8n-8k+7,$

$f(y_kx_{k+1})=8n-8k+2,\ f(x_kz_k)=8n-8k+8,\ f(x_kw_k)=8n-8k+5,$

$f(z_kx_{k+1})=8n-8k+4,\ f(w_kx_{k+1})=8n-8k+1,$

$f(z_ky_k)=8n-8k+6$, and $f(y_kw_k)=8n-8k+3.$

It is sufficient to prove that the k^{th} block of nW_4 is superior edge bimagic where $1\le k\le n$

$f(x_k)+f(y_k)+f(x_ky_k)=4k+8n-3+4k+8n-1+8n-8k+7=24n+3,$

$f(y_k)+f(x_{k+1})+f(y_kx_{k+1})=4k+8n-1+4k+8n+1+8n-8k+2=24n+2.$

Similarly for any remaining edge uv, $f(u)+f(v)+f(uv)$ equals $24n+3$ or $24n+2$. Hence the graph nW_4 admits superior edge bimagic labelling for $n\ge 1$. □

Theorem 2.4 *There exist a graph from the class $P_n\hat{o}K_{1,m}$ that admits superior edge magic labeling if n is odd and superior edge bimagic labelling if n is even.*

Proof Consider the path P_n with vertex set $\{x_1,x_2,\cdots,x_n\}$ and edge set $\{x_ix_{i+1};1\le i\le n-1\}$. Let $K_{1,m}$ have vertex set $\{y_1,y_2,\cdots,y_{m+1}\}$ and edge set $\{y_1y_j;2\le j\le m+1\}$.

Let G be a one of the graph from the class $P_n\hat{O}K_{1,m}$ where we superimpose on the vertex say y_1 of $K_{1,m}$ on the selected vertex x_n in P_n. The vertex set and edge set of graph G is given by $V=\{x_i,y_j;1\le i\le n,2\le j\le m+1\}$ and $E=E_1\cup E_2$, where $E_1=\{x_ix_{i+1};1\le i\le n-1\}$ and $E_2=\{x_ny_j;2\le j\le m-1\}$.

Case 1. n is odd.

Let $f:V\cup E\to\{1,2,3,\cdots,2m+2n-1\}$ be the bijective function defined by

$$f(x_i)=\begin{cases}(4m+4n-1-i)/2; & i=1,3,\cdots,n\\(4m+3n-1-i)/2; & i=2,4,\cdots,n-1\end{cases},$$

$f(y_j)=(2m+n+1-j);\ 2\le j\le m+1,$

$f(x_ix_{i+1})=i;\ 1\le i\le n-1,$

$f(x_n y_j) = (n - 2 + j);\ 2 \leq j \leq m + 1.$

For any edge $x_i x_{i+1}$ in E_1,

$$\begin{aligned} f(x_i) + f(x_{i+1}) + f(x_i x_{i+1}) &= \frac{4m + 4n - 1 - i + 4m + 3n - 1 - i - 1 + 2i}{2} \\ &= \frac{8m + 7n - 3}{2} = k. \end{aligned}$$

For any edge $x_n y_j$ in E_2,

$$\begin{aligned} f(x_n) + f(y_j) + f(x_n y_j) &= \frac{4m + 4n - 1 - n + 4m + 2n + 2 - 2j + 2n - 4 + 2j}{2} \\ &= \frac{8m + 7n - 3}{2} = k. \end{aligned}$$

Hence the graph G from the class $P_n \hat{O} K_{1,m}$ admits superior edge magic labeling if n is odd.

Case 2. n is even.

Let $f : V \cup E \to \{1, 2, 3, \cdots, 2m + 2n - 1\}$ be the bijective function defined by

$$f(x_i) = \begin{cases} (4m + 4n - 1 - i)/2; & i = 1, 3, \cdots, n - 1 \\ (4m + 3n - i)/2; & i = 2, 4, \ldots, n \end{cases},$$

$f(y_j) = (2m + n + 1 - j);\ 2 \leq j \leq m + 1,$

$f(x_i x_{i+1}) = i;\ 1 \leq i \leq n - 1,$

$f(x_n y_j) = (n - 2 + j);\ 2 \leq j \leq m + 1.$

For any edge $x_i x_{i+1}$ in E_1,

$$\begin{aligned} f(x_i) + f(x_{i+1}) + f(x_i x_{i+1}) &= \frac{4m + 4n - 1 - i + 4m + 3n - i - 1 + 2i}{2} \\ &= \frac{8m + 7n - 2}{2} = k_1. \end{aligned}$$

For any edge $x_n y_j$ in E_2,

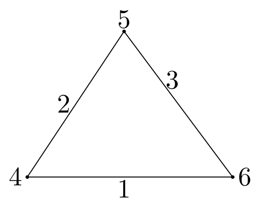
$$\begin{aligned} f(x_n) + f(y_j) + f(x_n y_j) &= \frac{4m + 3n - n + 4m + 2n + 2 - 2j + 2n - 4 + 2j}{2} \\ &= \frac{8m + 6n - 2}{2} = k_2. \end{aligned}$$

Hence the graph G from the class $P_n \hat{O} K_{1,m}$ admits superior edge bimagic labeling if n is even. $\square$

§3. Superior Edge Bimagic Labeling for Some General Graphs

Theorem 3.1 *A complete graph K_n $(n \geq 6)$ is not superior edge bimagic.*

Proof The superior edge bimagic labeling for the complete graph K_3, K_4 and K_5 are in the Figures 3-5 following.



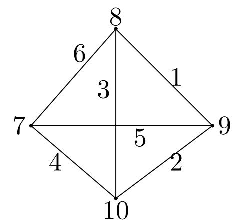


Figure 3 $k_1 = 11$, $k_2 = 14$ **Figure 4** $k_1 = 18$, $k_2 = 21$

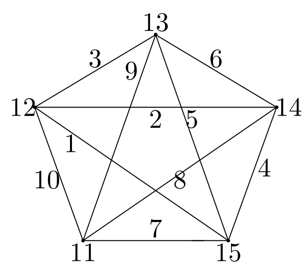


Figure 5 $k_1 = 28$, $k_2 = 33$

In case $n \geq 6$ we show that it is not to do super edge bimagic labeling. There are n vertices and nC_2 edges in a complete graph K_n. By labeling the n vertices from $(n^2 - n + 2)/2$, $(n^2 - n + 4)/2, \ldots, (n^2 + n)/2$ and adding for all edges uv in the complete graph K_n we find that vertex sum of edges

$$f(u) + f(v) = \begin{cases} n^2 - n + 3, n^2 - n + 4, \cdots, n^2 - n + 7, \cdots, n^2 + 1; \\ n^2 - n + 5, n^2 - n + 6, n^2 - n + 7, \cdots, n^2 + 2; \\ n^2 - n + 7, n^2 - n + 8, n^2 - n + 9, \cdots, n^2 + 3; \\ \cdots\cdots\cdots \\ n^2 - n - 3, n^2 - n - 2, n^2 - n - 1. \end{cases}$$

On observing the above sequence of vertex sums of the edges we find that starting from $n^2 - n + 7$ onwards each integers occurs at least three times. Hence on adding the label of edge $f(uv)$ to each sum $f(u) + f(v)$, it is impossible to obtain to common edge count k_1 or k_2. Hence the complete graph K_n for $n \geq 6$ is not superior edge bimagic labeling. □

Theorem 3.2 *If G has superior edge magic then $G + K_1$ admits edge bimagic total labelling.*

Proof Let $G(p, q)$ be a superior edge magic graph with bijective function $f : V \cup E \to$

$\{1, 2, 3, \cdots, p+q\}$ such that $f(u) + f(v) + f(uv) = k_1$ for all $uv \in E(G)$ where $f(E(G)) = \{1, 2, 3, \cdots, q\}$. Now we define a new graph $G_1 = G + K_1$ with vertex set $V_1 = V \cup \{x_1\}$ and edge set $E_1 = E \cup \{v_i x_1;\ 1 \le i \le p\}$.

Now define the bijective function $g : V_1 \cup E_1 \to \{1, 2, 3, \cdots, p+q, p+q+1, \cdots, 2p+q+1\}$ as $g(v) = f(v)$ for all $v \in V(G)$ with $f(v_i) = q + i$, $g(x_1) = p + q + 1$, $g(uv) = f(uv)$ for all $uv \in E(G)$ and $g(v_i x_1) = (2p + q + 2) - i;\ 1 \le i \le p$.

The edge set of G_1 consisting edges of G and remaining edges $\{v_i x_1; 1 \le i \le p\}$. Since G is superior edge magic, the edges of G have common count k_1. Now we need to prove edges $\{v_i x_1; 1 \le i \le p\}$ will have a common count k_2.

For edge $v_i x_1$, $g(v_i) + g(x_1) + g(v_i x_1) = q + i + p + q + 1 + (2p + q + 2) - i = 3p + 3q + 3 = k_2$. Hence $G + K_1$ admit edge bimagic total labelling. □

Theorem 3.3 *If G has superior edge magic then there exist a graph from the class $G\hat{O}P_n$ admits edge bimagic total labelling.*

Proof Let $G(p, q)$ be a superior edge magic graph with bijective function $f : V \cup E \to \{1, 2, 3, \cdots, p+q\}$ such that $f(u) + f(v) + f(uv) = k_1$ for all $uv \in E(G)$ where $f(E(G)) = \{1, 2, 3, \cdots, q\}$. Consider the graph P_n with Vertex set $\{x_1, x_2, \cdots, x_n\}$ and Edge set $\{x_i x_{i+1}; 1 \le i \le n-1\}$. We superimpose one of the vertex say x_1 of P_n on selected vertex v_p in G. Now we define new graph $G_1 = G\hat{O}P_n$ with vertex set $V_1 = V \cup \{x_i; 2 \le i \le n\}$ and edge set $E_1 = E \cup \{x_i x_{i+1}; 1 \le i \le n-1\}$.

Now define the bijective function $g : V_1 \cup E_1 \to \{1, 2, 3, \cdots, p+q, p+q+1, \cdots, p+q+2n-2\}$ as $g(v) = f(v)$ for all $v \in V(G)$ with $f(v_i) = q + i$, $g(uv) = f(uv)$ for all $uv \in E(G)$ if n is odd,

$$g(x_i) = \begin{cases} (2p + 2q + i - 1)/2; & i = 1, 3, 5, 7, \cdots, n \\ (2p + 2q + n - 1 + i)/2; & i = 2, 4, 6, \cdots, n-1 \end{cases},$$

$$g(x_i x_{i+1}) = (p + q + 2n - 1) - i; \quad 1 \le i \le n-1$$

if n is even and

$$g(x_i) = \begin{cases} (2p + 2q + i - 1)/2; & i = 1, 3, 5, 7, \cdots, n-1 \\ (2p + 2q + n - 2 + i)/2; & i = 2, 4, 6, \cdots, n \end{cases},$$

$$g(x_i x_{i+1}) = (p + q + 2n - 1) - i; \quad 1 \le i \le n-1.$$

The edge set of G_1 consisting edges of G and remaining edges $\{x_i x_{i+1}; 1 \le i \le n-1\}$. Since G is superior edge magic, the edges of G will have common count k_1. Now we need to prove edges $\{x_i x_{i+1}; 1 \le i \le n-1\}$ have a common count k_2. We prove it in two cases.

Case 1. n is odd.

For edge $x_i x_{i+1}$

$$\begin{aligned} g(x_i) + g(x_{i+1}) + g(x_i x_{i+1}) &= (2p + 2q + i - 1)/2 + (2p + 2q + n + i)/2 \\ &\quad + (p + q + 2n - 1) - i \\ &= (6p + 6q + 5n - 3)/2 = k_2 \end{aligned}$$

Thus we have $G\hat{O}P_n$ has two common count k_1 and k_2 if n is odd.

Case 2. n is even.

For edge $x_i x_{i+1}$

$$\begin{aligned} g(x_i) + g(x_{i+1}) + g(x_i x_{i+1}) &= (2p + 2q + i - 1)/2 + (2p + 2q + n - 1 + i)/2 \\ &\quad + (p + q + 2n - 1) - i \\ &= (6p + 6q + 5n - 4)/2 = k_2 \end{aligned}$$

Thus we have $G\hat{O}P_n$ has two common count k_1 and k_2 if n is even.

Hence there exist a graph from the class $G\hat{O}P_n$ admits edge bimagic total labelling. □

Theorem 3.4 *If G has superior edge magic then there exist a graph from the class $G\hat{O}K_{1,n}$ admits edge bimagic total labelling.*

Proof Let $G(p,q)$ be a superior edge magic graph with bijective function $f : V \cup E \to \{1, 2, 3, \cdots, p + q\}$ such that $f(u) + f(v) + f(uv) = k_1$ for all $uv \in E(G)$ where $f(E(G)) = \{1, 2, 3, \cdots, q\}$. Consider the graph $K_{1,n}$ with Vertex set $\{x_1, x_2, \cdots, x_{n+1}\}$ and Edge set $\{x_1 x_i; 2 \le i \le n + 1\}$. We superimpose one of the vertex say x_1 of $K_{1,n}$ on selected vertex v_p in G. Now we define new graph $G_1 = G\hat{O}K_{1,n}$ with vertex set $V_1 = V \cup \{x_i; 2 \le i \le n + 1\}$ and edge set $E_1 = E \cup \{x_1 x_i; 2 \le i \le n + 1\}$.

Now define the bijective function $g : V_1 \cup E_1 \to \{1, 2, 3, \cdots, p+q, p+q+1, \ldots, p+q+2n\}$ as $g(v) = f(v)$ for all $v \in V(G)$ with $f(v_i) = q + i$, $g(uv) = f(uv)$ for all $uv \in E(G)$, $g(x_i) = p + q - 1 + i$; $1 \le i \le n + 1$ and $g(x_1 x_i) = (p + q + 2n + 2) - i$; $2 \le i \le n + 1$.

The edge set of G_1 consisting edges of G and remaining edges $\{x_1 x_i; 2 \le i \le n+1\}$. Since G is superior edge magic, the edges of G will have common count k_1. Now we need to prove edges $\{x_1 x_i; 2 \le i \le n + 1\}$ will have a common count k_2. For edge $x_1 x_i$,

$$\begin{aligned} g(x_1) + g(x_i) + g(x_1 x_i) &= p + q + p + q - 1 + i + (p + q + 2n + 2) - i \\ &= 3p + 3q + 2n + 1 = k_2 \end{aligned}$$

Thus we have $G\hat{O}K_{1,n}$ has two common count k_1 and k_2. Hence there exist a graph from the class $G\hat{O}K_{1,n}$ admits edge bimagic total labelling. □

References

[1] J.A.Gallian, A dynamic survey of graph labeling, *Electronic Journal of Combinatorics*, 18,

DS6, (2011).

[2] N.Hartsfield and G. Ringel, *Pearls in Graph Theory*, Academic Press, San Diego, 1990.

[3] A.Kotzig and A. Rosa, Magic valuations of finite graphs, *Canada Math. Bull.*, Vol. 13, 1970, 451-461.

[4] F.Harary, *Graph Theory*, Addison Wesley, Reading, Massachusetts, 1969.

[5] W.D.Wallis, *Magic Graphs*, Birkhauser, Basel 2001.

[6] J.Baskar Babujee, Bimagic labeling in path graphs, *The Mathematics Education*, Volume 38, No.1, 2004, 12-16.

[7] J.Baskar Babujee, On edge bimagic labeling, *Journal of Combinatorics, Information & System Sciences*, Vol. 28-29, Nos. 1-4, (2004), 239-244.

[8] J.Baskar Babujee and S.Babitha, New constructions of edge bimagic graphs from magic graphs, *Applied Mathematics*, Vol. 2, No. 11, (2011), 1393-1396.

[9] J.Baskar Babujee, R.Jagadesh, Super edge bimagic labeling for Trees, *International Journal of Analyzing methods of Components and Combinatorial Biology in Mathematics*, Vol.1 No. 2, 2008, 107-116.

[10] J.Baskar Babujee, R.Jagadesh, Super edge bimagic labeling for graph with cycles, *Pacific-Asian Journal of Mathematics*, Volume 2, No. 1-2, 2008, 113-122.

Math.Combin.Book Ser. Vol.3(2015), 43-54

Spherical Images of Special Smarandache Curves in E^3

Vahide Bulut and Ali Caliskan

(Department of Mathematics, Ege University, Izmir, 35100, Turkey)

E-mail: vahidebulut@mail.ege.edu.tr, ali.caliskan@ege.edu.tr

Abstract: In this study, we introduce the spherical images of some special Smarandache curves according to Frenet frame and Darboux frame in E^3. Besides, we give some differential geometric properties of Smarandache curves and their spherical images.

Key Words: Smarandache curves, S.Frenet frame, Darboux frame, Spherical image.

AMS(2010): 53A04.

§1. Introduction

Curves especially regular curves are used in many fields such as CAGD, mechanics, kinematics and differential geometry. Researchers are used various curves in these fields. Special Smarandache curves are one of them. A regular curve in Minkowski spacetime, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve ([7]). Some authors have studied on special Smarandache curves ([1, 2, 7]).

In this paper, we give the spherical images of some special Smarandache curves according to Frenet frame and Darboux frame in E^3. Also, we give some relations between the arc length parameters of Smarandache curves and their spherical images.

§2. Preliminaries

Let $\alpha(s)$ be an unit speed curve that satisfies $||\alpha^{'}(s)|| = 1$ in E^3. S.Frenet frame of this curve in E^3 parameterized by arc length parameter s is,

$$\alpha^{'}(s) = T, \qquad \frac{T^{'}(s)}{||T^{'}(s)||} = N(s), \qquad T(s) \times N(s) = B(s),$$

where $T(s)$ is the unit tangent vector, $N(s)$ is the unit principal normal vector and $B(s)$ is the

[1]Received February 9, 2015, Accepted August 10, 2015.

unit binormal vector of the curve $\alpha(s)$. The derivative formulas of S.Frenet are,

$$\begin{bmatrix} T^{'} \\ N^{'} \\ B^{'} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & \tau \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{1}$$

where $\kappa = \kappa(s) = ||T^{'}(s)||$ and $\tau = \tau(s) = ||B^{'}(s)||$ are the curvature and the torsion of the curve $\alpha(s)$ at s, respectively [4].

Let S be a regular surface and a curve $\alpha(s)$ be on the surface S. Since the curve $\alpha(s)$ is also a space curve, the curve $\alpha(s)$ has S.Frenet frame as mentioned above. On the other hand, since the curve $\alpha(s)$ lies on the surface S, there exists another frame which is called Darboux frame $\{T, g, n\}$ of the curve $\alpha(s)$. T is the unit tangent vector of the curve $\alpha(s)$, n is the unit normal of the surface S and g is a unit vector given by $g = n \times T$.The derivative formulas of Darboux frame are

$$\begin{bmatrix} T^{'} \\ g^{'} \\ n^{'} \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ g \\ n \end{bmatrix}, \tag{2}$$

where, κ_g is the geodesic curvature, κ_n is the normal curvature and τ_g is the geodesic torsion of the curve $\alpha(s)$. The Darboux vector and the unit Darboux vector of this curve are given, respectively as follows

$$d = \tau_g T + \kappa_n g + \kappa_g n$$

$$c = \frac{d}{||d||} = \frac{\tau_g T + \kappa_n g + \kappa_g n}{\sqrt{\tau_g^2 + \kappa_n^2 + \kappa_g^2}}. \tag{3}$$

(1) $\alpha(s)$ is a geodesic curve if and only if κ_g=0.

(2) $\alpha(s)$ is an asymptotic line if and only if κ_n=0.

(3) $\alpha(s)$ is a principal line if and only if τ_g =0 ([6]).

The sphere in E^3 with the radius $r > 0$ and the center in the origin is defined by [3]

$$S^2 = \left\{ x = (x_1,\, x_2,\, x_3) \in E^3 : \langle x, x \rangle = r^2 \right\}.$$

Let the vectors of the moving frame of a curve $\alpha(s)$ with non-vanishing curvature are given. Assume that these vectors undergo a parallel displacement and become bound at the origin O of the Cartesian coordinate system in space. Then the terminal points of these vectors $T(s)$, $N(s)$ and $B(s)$ lie on the unit sphere S which are called the tangent indicatrix, the principal normal indicatrix and the binormal indicatrix, respectively of the curve $\alpha(s)$.

The linear elements ds_T, ds_N and ds_B of these indicatrices or spherical images can be easily obtained by means of (1). Since $T(s)$, $N(s)$ and $B(s)$ are the vector functions representing these

curves we find

$$\begin{cases} ds_T^2 = \kappa^2 ds^2, \\ ds_N^2 = \left(\kappa^2 + \tau^2\right) ds^2, \\ ds_B^2 = \tau^2 ds^2. \end{cases} \tag{4}$$

Curvature and torsion appear here as quotients of linear elements; choosing the orientation of the spherical image by the orientation of the curve $\alpha(s)$ we have from (4)

$$\kappa = \frac{ds_T}{ds}, \qquad |\tau| = \frac{ds_B}{ds}. \tag{5}$$

Moreover, from (5) we obtain the Equation of Lancret ([5])

$$ds_N^2 = ds_T^2 + ds_B^2. \tag{6}$$

§3. Special Smarandache Curves According to S.Frenet Frame In E^3

3.1 TN- Smarandache Curves

Let $\alpha(s)$ be a unit speed regular curve in E^3 and $\{T, N, B\}$ be its moving S.Frenet frame. A Smarandache TN curve is defined by ([1])

$$\beta\left(s^*\right) = \frac{1}{\sqrt{2}}\left(T + N\right). \tag{7}$$

Let moving S. Frenet frame of this curve be $\{T^*, N^*, B^*\}$.

3.1.1 Spherical Image of the Unit Vector T_β^*

We can find the relation between the arc length parameters ds^* and ds as follows

$$\frac{ds^*}{ds} = \sqrt{\frac{2\kappa^2 + \tau^2}{2}}. \tag{8}$$

From the equations (5) and (8) we have

$$ds^* = \frac{\sqrt{2\kappa^2 + \tau^2}}{\kappa\sqrt{2}} ds_T. \tag{9}$$

From the equation (5) we obtain the spherical image of the unit vector T_β^* as

$$\frac{ds_T^*}{ds^*} = \kappa^* = \frac{\sqrt{2}\sqrt{\delta^2 + \mu^2 + \eta^2}}{\left(\sqrt{2\kappa^2 + \tau^2}\right)^2}, \tag{10}$$

where

$$\kappa^* = \frac{\sqrt{2}\sqrt{\delta^2 + \mu^2 + \eta^2}}{\left(\sqrt{2\kappa^2 + \tau^2}\right)^2}. \tag{11}$$

Here ([1]),

$$\begin{cases} \delta = -\left[\kappa^2\left(2\kappa^2+\tau^2\right)+\tau\left(\tau\kappa^{'}-\kappa\tau^{'}\right)\right], \\ \mu = -\left[\kappa^2\left(2\kappa^2+3\tau^2\right)+\tau\left(\tau^3-\tau\kappa^{'}+\kappa\tau^{'}\right)\right], \\ \eta = \kappa\left[\tau\left(2\kappa^2+\tau^2\right)-2\left(\tau\kappa^{'}-\kappa\tau^{'}\right)\right]. \end{cases}$$

Then, from the equations (9) and (10)

$$ds_T^* = \frac{\sqrt{\delta^2+\mu^2+\eta^2}}{\kappa\left(2\kappa^2+\tau^2\right)^{3/2}}ds_T \tag{12}$$

is obtained.

3.1.2 Spherical Image of the Unit Vector N_β^*

If we use the equation (6) we have

$$\frac{ds_N^*}{ds^*} = \sqrt{\left(\kappa^*\right)^2+\left(\tau^*\right)^2}. \tag{13}$$

Besides, from the equations (6), (8) and (13)

$$ds_N^* = \sqrt{\left(\kappa^*\right)^2+\left(\tau^*\right)^2}\frac{\sqrt{2\kappa^2+\tau^2}}{\sqrt{2}\sqrt{\kappa^2+\tau^2}}ds_N \tag{14}$$

is obtained, where

$$\tau^* = \frac{\sqrt{2}\left[\left(\kappa^2+\tau^2-\kappa^{'}\right)\left(\kappa\sigma+\tau\omega\right)+\kappa\left(\kappa\tau+\tau^{'}\right)\left(\phi-\omega\right)+\left(\kappa^2+\kappa^{'}\right)\left(\kappa\sigma-\tau\phi\right)\right]}{\left[\tau\left(2\kappa^2+\tau^2\right)+\kappa^{'}\tau-\kappa\tau^{'}\right]^2+\left(\tau\kappa^{'}-\kappa\tau^{'}\right)^2+\left(2\kappa^3+\kappa\tau^2\right)^2} \tag{15}$$

and

$$\begin{cases} \omega = \kappa^3+\kappa\left(\tau^2-3\kappa^{'}\right)-\kappa^{''}, \\ \phi = -\kappa^3-\kappa\left(\tau^2+3\kappa^{'}\right)-3\tau\tau^{'}+\kappa^{''}, \\ \sigma = -\kappa^2\tau-\tau^3+2\tau\kappa^{'}+\kappa\tau^{'}+\tau^{''}. \end{cases}$$

3.1.3 Spherical Image of the Unit Vector B_β^*

From the equations (5) and (15) we have

$$\frac{ds_B^*}{ds^*} = \tau^*. \tag{16}$$

On the other hand, the following formula is found from the equations (5), (8) and (16).

$$ds_B^* = \tau^*\frac{\sqrt{2\kappa^2+\tau^2}}{\tau\sqrt{2}}ds_B \tag{17}$$

Example 1 Let the curve $\alpha(s) = \left(\frac{4}{5}\sin t,\, 2-\cos t,\, \frac{3}{5}\sin t\right)$ is given. TN-Smarandache curve

of this curve is found as

$$\beta\left(s^*\right)=\left[\frac{4}{5\sqrt{2}}\left(\cos t-\sin t\right),\ \frac{1}{\sqrt{2}}\left(\sin t+\cos t\right),\ \frac{3}{5\sqrt{2}}\left(\cos t-\sin t\right)\right].$$

The spherical images of T^*, N^* and B^* for the curve $\beta\left(s^*\right)$ are shown in Figures 1, 2 and 3, respectively.

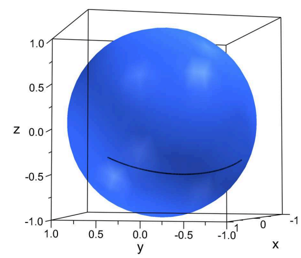


Figure 1 Spherical image of T^*

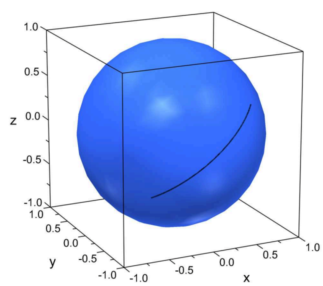


Figure 2 Spherical image of N^*

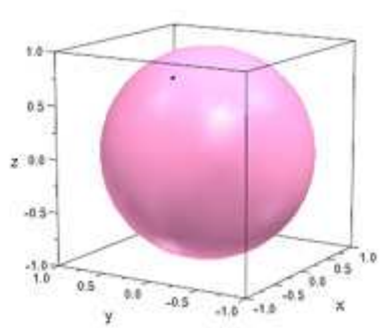


Figure 3 Spherical image of B^*

3.2 NB- Smarandache Curves

Let $\alpha(s)$ be a unit speed regular curve in E^3 and $\{T,N,B\}$ be its moving S. Frenet frame. Smarandache NB curve is defined by ([1])

$$\beta\left(s^*\right)=\frac{1}{\sqrt{2}}\left(N+B\right). \tag{18}$$

3.2.1 Spherical Image of the Unit Vector T_β^*

From the equations (5) and (8) we have

$$ds^* = \frac{\sqrt{2\kappa^2+\tau^2}}{\kappa\sqrt{2}} ds_T. \tag{19}$$

From the equation (5), we obtain the spherical image of the T_β^* as

$$\frac{ds_T^*}{ds^*} = \kappa^* = \frac{\sqrt{2}\sqrt{\gamma_1^2+\gamma_2^2+\gamma_3^2}}{\left(2\kappa^2+\tau^2\right)^2}, \tag{20}$$

where

$$\begin{cases} \gamma_1 = \kappa\tau\left(2\kappa+\tau^{'}\right)+\tau^2\left(\tau-\kappa^{'}\right), \\ \gamma_2 = -\left[\kappa^2\left(2\kappa^2+3\tau^2+2\tau^{'}\right)+\tau\left(\tau^3-2\kappa\kappa^{'}\right)\right], \\ \gamma_3 = 2\kappa^2\left(\tau^{'}-\tau^2\right)-\tau\left(\tau^3+2\kappa\kappa^{'}\right). \end{cases}$$

Then, the following formula is obtained from the equations (9) and (20).

$$ds_T^* = \frac{\sqrt{\gamma_1^2+\gamma_2^2+\gamma_3^2}}{\kappa\left(2\kappa^2+\tau^2\right)^{3/2}} ds_T \tag{21}$$

3.2.2 Spherical Image of the Unit Vector N_β^*

The spherical image of N_β^* can be found by using the equation (6) as

$$\frac{ds_N^*}{ds^*} = \sqrt{\left(\kappa^*\right)^2+\left(\tau^*\right)^2}, \tag{22}$$

where

$$\tau^* = \frac{\sqrt{2}\left(\kappa\varphi_3+\tau\varphi_1\right)\left(2\tau^2-\kappa^2\right)}{\left(2\tau^3-2\kappa^2\right)^2+\left(\kappa\tau'-\tau\kappa'\right)^2+\left(-\kappa^3+\kappa\tau'-\kappa'\tau\right)^2} \tag{23}$$

and

$$\begin{cases} \varphi_3 = -\tau^3-3\tau\tau^{'}+\kappa^2\tau+\tau^{''}, \\ \varphi_1 = -\kappa^3+\kappa\left(\tau^2+2\tau^{'}\right)+\tau\kappa^{'}-\kappa^{''}. \end{cases}$$

Besides, from the equations (6), (8) and (22)

$$ds_N^* = \sqrt{\left(\kappa^*\right)^2+\left(\tau^*\right)^2}\,\frac{\sqrt{2\kappa^2+\tau^2}}{\sqrt{2}\sqrt{\kappa^2+\tau^2}} ds_N \tag{24}$$

is obtained.

3.2.3 Spherical Image of the Unit Vector B_β^*

From the equations (5) and (23) we have

$$\frac{ds_B^*}{ds^*} = \tau^*. \tag{25}$$

On the other hand, from the equations (5), (8) and (25)

$$ds_B^* = \left[\frac{\sqrt{2}\left(\kappa\varphi_3 + \tau\varphi_1\right)\left(2\tau^2 - \kappa^2\right)}{\left(2\tau^3 - 2\kappa^2\right)^2 + \left(\kappa\tau' - \tau\kappa'\right)^2 + \left(-\kappa^3 + \kappa\tau' - \kappa'\tau\right)^2}\right]\frac{\sqrt{2\kappa^2 + \tau^2}}{\tau\sqrt{2}} ds_B \tag{26}$$

is found.

Example 2 Let the curve $\alpha(s) = \left(\frac{4}{5}\sin t,\, 2 - \cos t,\, \frac{3}{5}\sin t\right)$ is given. NB -Smarandache curve of this curve is

$$\beta(s^*) = \frac{1}{\sqrt{2}}\left[-\frac{4}{5}\sin t - \frac{3}{5},\, \cos t,\, -\frac{3}{5}\sin t + \frac{4}{5}\right].$$

The spherical images of T^* and N^* for the curve $\beta(s^*)$ are shown in Figures 4 and 5, respectively.

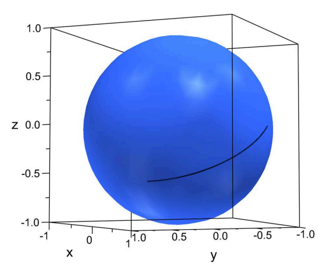


Figure 4 Spherical image of T^*

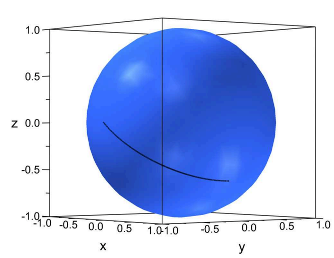


Figure 5 Spherical image of N^*

The spherical image of B^* for the curve $\beta(s^*)$ is a point similar to the Figure 3.

3.3 TB- Smarandache Curves

Let $\alpha(s)$ be a unit speed regular curve in E^3 and $\{T, N, B\}$ be its moving S.Frenet frame. Smarandache TB curve is defined by ([1])

$$\beta(s^*) = \frac{1}{\sqrt{2}}(T + B). \tag{27}$$

3.3.1 Spherical Image of the Unit Vector T_β^*

We can find the spherical image of T_β^* from the equation (5) and obtain

$$\frac{ds_T^*}{ds^*} = \kappa^* = \frac{\sqrt{2}\sqrt{\sigma_1^2+\sigma_2^2+\sigma_3^2}}{\left(2\kappa^2+\tau^2\right)^2}, \tag{28}$$

where

$$\begin{cases} \sigma_1 = \left(2\kappa^2+\tau^2\right)\left(\kappa\tau-\kappa^2\right), \\ \sigma_2 = \left(2\kappa+\tau\right)\left(\kappa^{'}\tau-\kappa\tau^{'}\right), \\ \sigma_3 = \left(2\kappa^2+\tau^2\right)\left(\kappa\tau-\tau^2\right). \end{cases}$$

From the equations (5) and (8)

$$ds^* = \frac{\sqrt{2\kappa^2+\tau^2}}{\kappa\sqrt{2}} ds_T \tag{29}$$

is obtained. Then, the formula following is acquired from equations (28) and (29).

$$ds_T^* = \frac{\sqrt{\sigma_1^2+\sigma_2^2+\sigma_3^2}}{\kappa\left(2\kappa^2+\tau^2\right)^{3/2}} ds_T \tag{30}$$

3.3.2 Spherical Image of the Unit Vector N_β^*

If we use the equation (6) we have the spherical image of N_β^* as

$$\frac{ds_N^*}{ds^*} = \sqrt{\left(\kappa^*\right)^2+\left(\tau^*\right)^2}. \tag{31}$$

Besides, from the equations (6), (8) and (31)

$$ds_N^* = \sqrt{\left(\kappa^*\right)^2+\left(\tau^*\right)^2}\,\frac{\sqrt{2\kappa^2+\tau^2}}{\sqrt{2}\sqrt{\kappa^2+\tau^2}} ds_N \tag{32}$$

is obtained, where

$$\tau^* = \frac{\sqrt{2}\left(\tau-\kappa\right)^2\left(\kappa\Phi_3+\tau\Phi_1\right)}{\left[\tau\left(\kappa-\tau\right)^2\right]^2+\left[\kappa\left(\kappa-\tau\right)^2\right]^2}, \tag{33}$$

$$\begin{cases} \Phi_3 = 2\tau\left(\kappa^{'}-\tau^{'}\right)+\tau^{'}\left(\kappa-\tau\right), \\ \Phi_1 = \kappa^{'}\left(\tau-\kappa\right)+2\kappa\left(\tau^{'}-\kappa^{'}\right). \end{cases}$$

3.3.3 Spherical Image of the Unit Vector B_β^*

From the equations (5) and (33) we have

$$\frac{ds_B^*}{ds^*} = \tau^*. \tag{34}$$

On the other hand, the following formula is found from the equations (5), (8) and (34).

$$ds_B^* = \left[\frac{\sqrt{2}\left(\tau-\kappa\right)^2\left(\kappa\Phi_3+\tau\Phi_1\right)}{\left[\tau\left(\kappa-\tau\right)^2\right]^2+\left[\kappa\left(\kappa-\tau\right)^2\right]^2}\right]\frac{\sqrt{2\kappa^2+\tau^2}}{\tau\sqrt{2}}ds_B \tag{35}$$

Example 3 Let the curve $\alpha\left(s\right)=\left(\frac{4}{5}\sin t,\, 2-\cos t,\, \frac{3}{5}\sin t\right)$ is given. TB -Smarandache curve of this curve is

$$\beta\left(s^*\right)=\frac{1}{\sqrt{2}}\left[\frac{4}{5}\cos t-\frac{3}{5},\, \sin t,\, \frac{3}{5}\cos t+\frac{4}{5}\right].$$

The spherical images of T^* and N^* for the curve $\beta\left(s^*\right)$ are shown in Figures 6 and 7, respectively.

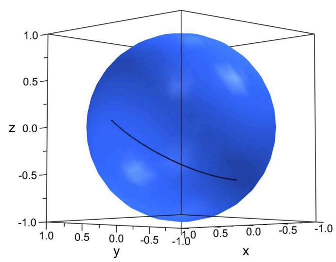


Figure 6 Spherical image of T^*

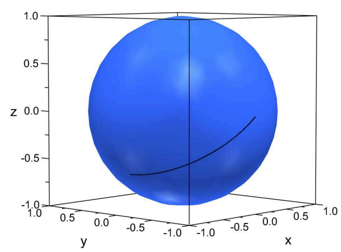


Figure 7 Spherical image of N^*

The spherical image of B^* for the curve $\beta\left(s^*\right)$ is a point similar to the Figure 3.

3.4 TNB- Smarandache Curves

Let $\alpha(s)$ be a unit speed regular curve in E^3 and $\{T, N, B\}$ be its moving S.Frenet frame. Smarandache TNB curve is defined by ([1])

$$\beta\left(s^*\right)=\frac{1}{\sqrt{3}}\left(T+N+B\right). \tag{36}$$

Remark 1 The spherical images of the curve $\beta\left(s^*\right)$ can be found in a similar way as presented above.

§4. Spherical Images of Darboux Frame $\{T, g, n\}$

Let S be an oriented surface in E^3. Let $\alpha(s)$ be a unit speed regular curves in E^3 and $\{T, g, n\}$ be Darboux frame of this curve.

4.1 Spherical Image of The Unit Vector T

The differential geometric properties of the spherical image of the unit vector T are given as

$$\frac{dT}{ds_T} = \frac{dT}{ds}.\frac{ds}{ds_T}$$

$$\frac{dT}{ds_T} = (\kappa_g g + \kappa_n n).\frac{ds}{ds_T}$$

$$\frac{ds_T}{ds} = \sqrt{\kappa_g^2 + \kappa_n^2}. \tag{37}$$

On the other hand, from (4) and (37)

$$\kappa = \sqrt{\kappa_g^2 + \kappa_n^2}. \tag{38}$$

can be written.

4.2 Spherical Image of The Unit Vector g

The differential geometric properties of the spherical image of the unit vector g are found as

$$\frac{dg}{ds_g} = \frac{dg}{ds}.\frac{ds}{ds_g}$$

$$\frac{dg}{ds_g} = (-\kappa_g T + \tau_g n).\frac{ds}{ds_g}$$

The relation between the arc length parameters are given as follows

$$\frac{ds_g}{ds} = \sqrt{\kappa_g^2 + \tau_g^2}. \tag{39}$$

4.3 Spherical Image of The Unit Vector n

The differential geometric properties of the spherical image of the unit vector n are given as

$$\frac{dn}{ds_n} = \frac{dn}{ds}.\frac{ds}{ds_n}$$

$$\frac{dn}{ds_n} = (-\kappa_n T - \tau_g g).\frac{ds}{ds_n}$$

Also, the relation between the arc length parameters is obtained as

$$\frac{ds_n}{ds} = \sqrt{\kappa_n^2 + \tau_g^2}. \tag{40}$$

Results:

i) *If $\alpha(s)$ is a geodesic curve, for $\kappa_g = 0$,*

$$\frac{ds_T}{ds} = \kappa_n = \kappa, \qquad \tfrac{ds_g}{ds} = \tau_g, \qquad \frac{ds_n}{ds} = \sqrt{\kappa_n^2 + \tau_g^2} = \sqrt{\kappa^2 + \tau_g^2}.$$

Also, the unit Darboux vector is as follows

$$c = \frac{\tau_g T + \kappa_n g}{\sqrt{\tau_g^2 + \kappa_n^2}}.$$

ii) *If $\alpha(s)$ is an asymptotic line, for κ_n=0*

$$\frac{ds_T}{ds} = \kappa_g = \kappa, \qquad \tfrac{ds_g}{ds} = \sqrt{\kappa_g^2 + \tau_g^2} = \sqrt{\kappa^2 + \tau_g^2}, \qquad \frac{ds_n}{ds} = \tau_g,$$

and the unit Darboux vector is

$$c = \frac{\tau_g T + \kappa_g n}{\sqrt{\tau_g^2 + \kappa_g^2}}.$$

iii) *If $\alpha(s)$ is a line of curvature, for τ_g=0*

$$\frac{ds_T}{ds} = \sqrt{\kappa_g^2 + \kappa_n^2} = \kappa, \qquad \tfrac{ds_g}{ds} = \kappa_g, \qquad \frac{ds_n}{ds} = \kappa_n,$$

and the unit Darboux vector is

$$c = \frac{\kappa_g n + \kappa_n g}{\sqrt{\kappa_g^2 + \kappa_n^2}}.$$

§5. Special Smarandache Curves According To Darboux Frame In E^3

5.1 Tg- Smarandache Curves

Let S be an oriented surface in E^3. Let $\alpha(s)$ be a unit speed regular curve in E^3, $\{T, N, B\}$ and $\{T, g, n\}$ be its S.Frenet frame and Darboux frame, respectively. Smarandache Tg curve is defined by

$$\beta(s^*) = \frac{1}{\sqrt{2}}(T + g). \tag{41}$$

4.2 Tn- Smarandache Curves

Let S be an oriented surface in E^3. Let $\alpha(s)$ be a unit speed regular curve in E^3, $\{T, N, B\}$

and $\{T, g, n\}$ be its S.Frenet frame and Darboux frame, respectively. Smarandache Tn curve is defined by

$$\beta(s^*) = \frac{1}{\sqrt{2}}(T+n). \tag{42}$$

4.3 gn- Smarandache Curves

Let S be an oriented surface in E^3. Let $\alpha(s)$ be a unit speed regular curve in E^3, $\{T, N, B\}$ and $\{T, g, n\}$ be its S.Frenet frame and Darboux frame, respectively. Smarandache gn curve is defined by

$$\beta(s^*) = \frac{1}{\sqrt{2}}(g+n). \tag{43}$$

4.4 Tgn- Smarandache Curves

Let S be an oriented surface in E^3. Let $\alpha(s)$ be a unit speed regular curve in E^3, $\{T, N, B\}$ and $\{T, g, n\}$ be its S.Frenet frame and Darboux frame, respectively. Smarandache Tgn curve is defined by

$$\beta(s^*) = \frac{1}{\sqrt{3}}(T+g+n). \tag{44}$$

(See [2].)

Remark 2 The spherical images of these curves can be easily obtained by the similar way as explained in Section 4.

§6. Conclusion

Spherical mechanisms are very important for robotics. Spherical curves which are drawn by spherical mechanisms are used widely in kinematics and robotics. For this purpose, we presented the spherical images of special Smarandache curves and obtained some relations between them.

References

[1] A.T.Ali, Special Smarandache curves in the Euclidean space, *International Journal of Mathematical Combinatorics*, Vol.2(2010),30-36.

[2] O.Bektas and S.Yuce, Special Smarandache curves according to Darboux frame in Euclidean 3- space, *arXiv*: 1203. 4830, 1, 2012.

[3] M.P.Do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice Hall, Englewood Cliffs, NJ, 1976.

[4] H.Guggenheimer, *Diffrential Geometry*, McGraw-Hill Book Company, 1963.

[5] E.Kreyszig, *Differential Geometry*, Dover Publications, 1991.

[6] B.O'Neill, *Elemantery Differential Geometry*, Academic press Inc. New York, 1966.

[7] M.Turgut and S.Yilmaz, Smarandache curves in Minkowski space-time, *International Journal of Mathematical Combinatorics*, Vol.3(2008), 51-55.

Math.Combin.Book Ser. Vol.3(2015), 55-61

Variations of Orthogonality of Latin Squares

Vadiraja Bhatta G.R.

Department of Mathematics

Manipal Institute of Technology, Manipal University, Manipal, Karnataka-576104, India

B.R.Shankar

Department of Mathematical and Computational Sciences

National Institute of Technology, Surathkal, Karnataka, India

E-mail: vadiraja.bhatta@manipal.edu

Abstract: Orthogonal properties of Latin squares represented by permutation polynomials are discussed. Pairs of bivariate polynomials over small rings are considered.

Key Words: Bivariate polynomials, Latin squares, orthogonal Latin squares

AMS(2010): 08B15.

§1. Introduction

1.1 Latin Squares

Combinatorial theory is one of the fastest growing areas of modern mathematics. Combinatorial designs have wide applications in various fields, including coding theory and cryptography. Many examples of combinatorial designs can be listed like linked design, balanced design, one-factorization, graph etc. *Latin square* is one such combinatorial concept.

Definition 1.1 *A Latin square of order (or size) n is an $n \times n$ array based on some set S of n symbols (treatments), with the property that every row and every column contains every symbol exactly once.*

In other words, every row and every column is a permutation of S. Also it can be thought of as a two dimensional analogue of a permutation. A Latin square can be viewed as a quadruple $(R, C, S; L)$, where R, C, and S are sets of cardinality n, L is a mapping $L : R \times C \to S$ such that for any $i \in R$ and $x \in S$, the equation

$$L(i, j) = x$$

has a unique solution $j \in C$, and for any $j \in C$, $x \in S$, the same equation has a unique solution

[1]Received December 21, 2014, Accepted August 12, 2015.

$i \in R$. That is any two of $i \in R, j \in C, x \in S$ uniquely determine the third so that $L(i,j) = x$. i.e., the cell in row i and column j contains the symbol $L(i,j)$.

Using the concept of permutation functions we can define Latin squares as follows:

Definition 1.2 *A function $f : S^2 \to S$ on a finite set S of size $n > 1$ is said to be a Latin square (of order n) if for any $a \in S$ both the functions $f(a,.)$ and $f(.,a)$ are permutations of S.*

Here, $f(a,.)$ determines the rows and $f(.,a)$ determines the columns of the Latin square. Latin squares exist for all n, as an obvious example we can consider addition modulo n.

Example 1.3 A Latin square of order 5 over the set $\{a,b,c,d,e\}$ is below:

a	b	c	d	e
b	a	e	c	d
c	d	b	e	a
d	e	a	b	c
e	c	d	a	b

The terminology 'Latin square' originated with Euler who used a set of Latin letters for the set S. We discussed some results in [2] about the formation of Latin squares using bivariate permutation polynomials.

1.2 Orthogonal Latin Squares

One of the origins of the study of Latin squares is usually identified with the now famous problem of Euler concerning the arrangement of 36 officers of 6 different ranks and 6 different regiments into a 6×6 square. If there were such an officer holding each rank from each regiment, Euler's problem was to find an arrangement in which each rank and each regiment would be represented in every row and column. A solution requires two Latin squares of order 6; in one, the symbols represent the ranks and in the other regiments. Furthermore,the two Latin squares must be compatible in a very precise sense so that if one is superimposed on the other, each ordered pair occurs exactly once. Two such squares which exhibit such "compatibility" are *orthogonal Latin squares*. It is interesting to compare two Latin squares on the same set in different ways like *equivalence*, which we already saw above. *Orthogonality* is a very useful concept in the study of Latin squares, having a lot of applications in cryptography.

Definition 1.4 *Two Latin squares $L_1 : R \times C \to S$ and $L_2 : R \times C \to S$ (with the same row and column sets) are said to be orthogonal when for each ordered pair $(s,t) \in S \times T$, there is a unique cell $(x,y) \in R \times C$ so that*

$$L_1(x,y) = s \text{ and } L_2(x,y) = t.$$

That is, two Latin squares A and B of the same order n are *orthogonal*, if the n^2 ordered pairs (a_{ij}, b_{ij}), the pairs formed by superimposing one square on the other, are all different. One can say "*A is orthogonal to B*" or "*B is orthogonal to A*". So, the relation of orthogonality

is symmetric. In general, one can speak of k *mutually orthogonal* Latin squares $A_1, A_2, \cdots, A_k$ such that A_i is orthogonal to A_j whenever $i \neq j$.

§2. Orthogonality and Bivariate Polynomials

Using the concept of permutation functions orthogonality can be defined as follows:

Definition 2.1 *A pair of functions $f_1(*,*)$, $f_2(*,*)$ is said to be* ***orthogonal*** *if the pairs $(f_1(x,y), f_2(x,y))$ are all distinct, as x and y vary.*

Shannon observed that Latin squares are useful in cryptography. Schnorr and Vaudenay applied pairs of orthogonal Latin squares(which they called *multipermutations*) to cryptography. The following theorem is due to Rivest [1]:

Theorem 2.2 *There are no two polynomials $P_1(x,y)$, $P_2(x,y)$ modulo 2^w for $w \geq 1$ that form a pair of orthogonal Latin squares.*

In fact Euler believed that no pair of orthogonal Latin squares of order 6 exist, and this was not shown until Tarry did so around 1900 by means of an almost exhaustive search. While there are other less tedious methods now available to prove this result, it is still not an easy task to prove this without the aid of a computer [3]. Euler's actual conjecture was far stronger in that he speculated that there did not exist orthogonal Latin squares for orders $n = 6$, 10, 14,$\cdots$. This famous conjecture, which is associated with Euler's name remained unsolved until Bose, Parker and Shrikhande showed it to be false for $n = 10$, 14,$\cdots$. in a series of papers in 1959 and 1960 ([4]).

If we have two orthogonal Latin squares of order 4, both over the set, $\{1,2,3,4\}$, the configuration of their superposition is as follows:

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

The orthogonal configuration is:

(1,1)	(2,2)	(3,3)	(4,4)
(2,3)	(1,4)	(4,1)	(3,2)
(3,4)	(4,3)	(1,2)	(2,1)
(4,2)	(3,1)	(2,4)	(1,3)

Rivest [1] proved that no two bivariate polynomials modulo 2^w, for $w \geq 1$ can form a pair of orthogonal Latin squares. This is because all the bivariate polynomials over Z_n, where $n = 2^w$, will form Latin squares which can be equally divided into 4 parts as shown below, where the $n/2 \times n/2$ squares A and D are identical and $n/2 \times n/2$ squares B and C are identical.

A	B
C	D

So, no two such Latin squares can be orthogonal.

Theorem 2.3 *There are no two bivariate polynomials $P_1(x,y)$ and $P_2(x,y)$ over Z_n, where n is even, which can form orthogonal Latin squares.*

Proof Let $n = 2m$ and $Q(x)$ be any univariate polynomial over Z_n.Then, $Q(x+m) = Q(x) + m(mod\ n)$ for all $x \in Z_n$. Hence,

$$\begin{aligned} P_1(x+m, y+m) &= P_1(x, y+m) + m\ (mod\ n) \\ &= P_1(x,y) + 2m\ (mod\ n) \\ &= P_1(x,y)\ (mod\ n) \end{aligned}$$

The same holds for $P_2(x,y)$ too. Therefore,$(P_1(x,y), P_2(x,y) = P_1(x+m, y+m), P_2(x+m, y+m))$. So, the pair of Latin squares cannot be orthogonal. □

We do have examples of bivariate polynomials modulo $n \neq 2^w$ and n odd, such that the resulting Latin squares are orthogonal.

Example 2.4 The following is a pair of Latin squares over Z_9 which are orthogonal to each other.

0	8	4	6	5	1	3	2	7
7	0	8	4	6	5	1	3	2
8	4	6	5	1	3	2	7	0
3	2	7	0	8	4	6	5	1
1	3	2	7	0	8	4	6	5
2	7	0	8	4	6	5	1	3
6	5	1	3	2	7	0	8	4
4	6	5	1	3	2	7	0	8
5	1	3	2	7	0	8	4	6

Latin square formed by

$5x + y + 3xy + 3x^2 + 6y^2$

0	5	7	6	2	4	3	8	1
2	4	3	8	1	0	5	7	6
7	6	2	4	3	8	1	0	5
6	2	4	3	8	1	0	5	7
8	1	0	5	7	6	2	4	3
4	3	8	1	0	5	7	6	2
3	8	1	0	5	7	6	2	4
5	7	6	2	4	3	8	1	0
1	0	5	7	6	2	4	3	8

Latin square formed by

$2x + 5y + 6xy + 3x^2 + 6y^2$

The two bivariate quadratic polynomials $x+5y+3xy+6x^2+3y^2$ and $4x+7y+6xy+3x^2+6y^2$ give two orthogonal Latin squares over Z_9. Also, $x + 4y + 3xy$ is a quadratic bivariate which gives a Latin square orthogonal to Latin square formed by $x + 5y + 3xy + 6x^2 + 3y^2$ over Z_9, but not to that of $4x + 7y + 6xy + 3x^2 + 6y^2$.

Remark 2.5 We have found many examples in which the rows or columns of the Latin square formed by quadratic bivariates over Z_n are cyclic shifts of a single permutation of $\{0, 1, 2, \cdots, n-1\}$. If two bivariates give such Latin squares, then corresponding to any one entry in one Latin square, if there are n different entries in n rows of the other Latin square, then those two Latin squares will be orthogonal.For instance, in the example below, the entries in the second square corresponding to the entry 0 in the first square are 0, 8, 7, 6, 5, 4, 3, 2, 1. The rows of the first square are all cyclic shifts of the permutation (0, 8, 4, 6, 5, 1, 3, 2, 7), not in order. Also the columns of the second square are the cyclic shifts of the permutation (0, 7, 2, 3, 1, 5, 6, 4, 8), not in order.

Example 2.6

0	8	4	6	5	1	3	2	7
7	0	8	4	6	5	1	3	2
8	4	6	5	1	3	2	7	0
3	2	7	0	8	4	6	5	1
1	3	2	7	0	8	4	6	5
2	7	0	8	4	6	5	1	3
6	5	1	3	2	7	0	8	4
4	6	5	1	3	2	7	0	8
5	1	3	2	7	0	8	4	6

Latin square formed by

$5x + y + 3xy + 3x^2 + 6y^2$

0	4	2	3	7	5	6	1	8
7	8	3	1	2	6	4	5	0
2	0	1	5	3	4	8	6	7
3	7	5	6	1	8	0	4	2
1	2	6	4	5	0	7	8	3
5	3	4	8	6	7	2	0	1
6	1	8	0	4	2	3	7	5
4	5	0	7	8	3	1	2	6
8	6	7	2	0	1	5	3	4

Latin square formed by

$7x + 4y + 6xy + 6x^2 + 3y^2$

Instead of looking for an orthogonal mate of Latin square formed by some other polynomial we looked at the mirror image of the square itself.

Example 2.7 The Latin square formed by the polynomial $4x + 7y + 6xy + 6x^2 + 3y^2$ over Z_9 and its mirror image are given below:

0	1	5	3	4	8	6	7	2
1	8	0	4	2	3	7	5	6
8	3	1	2	6	4	5	0	7
3	4	8	6	7	2	0	1	5
4	2	3	7	5	6	1	8	0
2	6	4	5	0	7	8	3	1
6	7	2	0	1	5	3	4	8
7	5	6	1	8	0	4	2	3
5	0	7	8	3	1	2	6	4

2	7	6	8	4	3	5	1	0
6	5	7	3	2	4	0	8	1
7	0	5	4	6	2	1	3	8
5	1	0	2	7	6	8	4	3
0	8	1	6	5	7	3	2	4
1	3	8	7	0	5	4	6	2
8	4	3	5	1	0	2	7	6
3	2	4	0	8	1	6	5	7
4	6	2	1	3	8	7	0	5

These two are orthogonal to each other. Over the ring Z_7, here is a Latin square formed by the polynomial $3x+4y$, with its mirror image, which are orthogonal to each other:

0	3	6	2	5	1	4
4	0	3	6	2	5	1
1	4	0	3	6	2	5
5	1	4	0	3	6	2
2	5	1	4	0	3	6
6	2	5	1	4	0	3
3	6	2	5	1	4	0

4	1	5	2	6	3	0
1	5	2	6	3	0	4
5	2	6	3	0	4	1
2	6	3	0	4	1	5
6	3	0	4	1	5	2
3	0	4	1	5	2	6
0	4	1	5	2	6	3

But this is not always true. We have the example below 2.8: in the ring Z_8, the Latin square formed by the polynomial $x+5y+4xy+2x^2+6y^2$ is not orthogonal with its mirror image.

Example 2.8

0	3	2	5	4	7	6	1
3	2	5	4	7	6	1	0
2	5	4	7	6	1	0	3
5	4	7	6	1	0	3	2
4	7	6	1	0	3	2	5
7	6	1	0	3	2	5	4
6	1	0	3	2	5	4	7
1	0	3	2	5	4	7	6

1	6	7	4	5	2	3	0
0	1	6	7	4	5	2	3
3	0	1	6	7	4	5	2
2	3	0	1	6	7	4	5
5	2	3	0	1	6	7	4
4	5	2	3	0	1	6	7
7	4	5	2	3	0	1	6
6	7	4	5	2	3	0	1

Here there are 32 distinct pairs, each appearing twice. Also all the pairs of the form (a, b), where one of a and b is odd and the other is even are appearing and all pairs that appear are of this form.

Theorem 2.9 *For odd n, Latin square over Z_n formed by a bivariate permutation polynomial $P(x, y)$ is orthogonal with its mirror image.*

Proof If $P(x, y)$ is a bivariate linear polynomial, then clearly the rows (columns) are simply cyclic shifts of a *single* row (column). Hence the Latin Squares got by $P(x, y)$ and its mirror image are orthogonal. In the general case, corresponding to the cell index (x, y), the index in the mirror image is $(n-1-x, y)$. Thus if L denotes the Latin square formed by $P(x, y)$ and L' is its mirror image, on superimposing L and L' the entry in cell index (x, y) will be $(P(x, y), P(n-1-x, y))$. Since P is a permutation polynomial, these pairs will all be distinct as x and y vary in Z_n. Hence they are orthogonal. □

Remark 2.10 In case of rings Z_n, where n is even, we know from Theorem 2.3, the Latin squares formed by bivariate permutation polynomials have four parts with diagonally opposite pair of parts being same. Mirror images of such squares are also of the same kind. So, in the

first $n/2$ rows we can get $n/2$ distinct pairs of corresponding entries. In the last $n/2$ rows, these pairs will repeat in the same order. So, these two pair of squares are not orthogonal.

§3. Conclusion

Identifying a pair of bivariate polynomials modulo n which represent a pair of orthogonal Latin squares is not obvious. But for odd n, a Latin square formed by a bivariate polynomial is orthogonal to its mirror image. Moreover, no two bivariate polynomials over Z_n, when n is even can form orthogonal Latin squares.

References

[1] Rivest R. L., Permutation Polynomials modulo 2^w, *Finite Fields and Their applications*, 7 (2001) 287-292.

[2] Vadiraja Bhatta and Shankar B. R., Permutation Polynomials modulo n, $n \neq 2^w$ and Latin Squares, *International Journal Of Mathematical Combinatorics,* Vol. 2 (2009) 58-65.

[3] Laywine, C.F. and Mullen, G.L., Discrete Mathematics using Latin squares, *John Wiley and Sons, Inc.* (1998).

[4] Lint J.V., and Wilson R. M., *A Course in Combinatorics*, Cambridge University Press, Cambridge, second edition, 2001.

Math.Combin.Book Ser. Vol.3(2015), 62-72

The Minimum Equitable Domination Energy of a Graph

P.Rajendra and R.Rangarajan

(Department of Studies in Mathematics, University of Mysore, Manasagangotri, Mysore - 570 006, India)

E-mail: prajumaths@gmail.com, rajra63@gmail.com

Abstract: A subset D of V is called an equitable dominating set [8] if for every $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leq 1$, where $deg(u)$ denotes the degree of vertex u and $deg(v)$ denotes the degree of vertex v. Recently, The minimum covering energy $E_c(G)$ of a graph is introduced by Prof. C. Adiga, and co-authors [1]. Motivated by [1], in this paper we define energy of minimum equitable domination $E_{ED}(G)$ of some graphs and we obtain bounds on $E_{ED}(G)$. We also obtain the minimum equitable domination determinant of some graph G given by $det_{ED}(G) = \mu_1\mu_2 \ldots \mu_n$ where $\mu_1, \mu_2, \ldots, \mu_n$ are eigenvalues of $A_{ED}(G)$.

Key Words: Minimum equitable domination set, spectrum of minimum equitable domination matrix, energy of minimum equitable domination, determinant of minimum equitable domination matrix.

AMS(2010): 05C50.

§1. Introduction

The energy of a graph and its applications to Organic Chemistry are given in detail in two important works by I. Gutman and co-authors [5, 9]. For more details with applications on the energy of a graph, one may refer [2, 4, 6, 9]. Recently, the minimum covering energy $E_c(G)$ of a graph is introduced by Prof. C. Adiga, and co-authors [1]. Motivated by [1], in this paper we define energy of minimum equitable domination $E_{ED}(G)$ of some graphs and we obtain bounds on $E_{ED}(G)$. We also obtain the minimum equitable domination determinant of some graph G given by $det_{ED}(G) = \mu_1\mu_2 \ldots \mu_n$ where $\mu_1, \mu_2, \ldots, \mu_n$ are eigenvalues of $A_{ED}(G)$.

Let G be a graph with set of vertices , $V = \{v_1, v_2, \cdots, v_n\}$ and set of edges, E. For a simple graph, i.e a graph without loops, multiple or directed edges, a subset D of V is called an equitable dominating set [8] if for every $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leq 1$, where $deg(u)$ denotes the degree of vertex u and $deg(v)$ denotes the degree of vertex v. Let ED is minimum equitable domination set of a graph G.

The minimum equitable domination matrix is defined as a square matrix $A_{ED}(G) = (a_{ij})$,

[1]Received December 9, 2014, Accepted August 15, 2015.

where

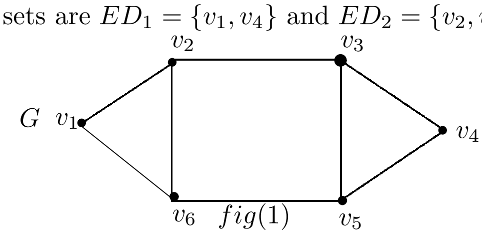
$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E \\ 1 & \text{if } i = j \text{ and } v_i \in ED \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The eigenvalues of the minimum equitable domination matrix $A_{ED}(G)$ are $\mu_1, \mu_2, \cdots, \mu_n$. Since the minimum equitable domination matrix is symmetric, its eigenvalues are real and can be written as $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. The energy of minimum equitable domination of a graph is defined as

$$E_{ED}(G) = \sum_{i=1}^{n} |\mu_i|. \tag{2}$$

We also obtain the minimum equitable domination determinant of some graph G given by $det_{ED}(G) = \mu_1 \mu_2 \cdots \mu_n$ where $\mu_1,\ \mu_2, \ldots, \mu_n$ are eigenvalues of $A_{ED}(G)$.

Example 1.1 The figure 1 shows the graph G with vertices $\{v_1, v_2, v_3, v_4, v_5, v_6\}$. Then minimum equitable domination sets are $ED_1 = \{v_1, v_4\}$ and $ED_2 = \{v_2, v_5\}$,



$fig(1)$

$$A_{ED_1}(G) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

characteristic polynomial of $A_{ED_1}(G)$ is $\Phi_6(G, \mu) = \mu^6 - 2\mu^5 - 7\mu^4 + 8\mu^3 + 12\mu^2$, the spectrum of $A_{ED_1}(G)$ is

$$Spec_{ED_1} = \begin{pmatrix} 3 & 2 & 0 & -1 & -2 \\ 1 & 1 & 2 & 1 & 1 \end{pmatrix}$$

and the energy of minimum equitable domination of ED_1 is $E_{ED_1} = 8$ and $det_{ED_1}(G) = 0$..

$$A_{ED_2}(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

characteristic polynomial of $A_{ED_2}(G)$ is $\Phi_6(G,\mu) = \mu^6 - 2\mu^5 - 7\mu^4 + 6\mu^3 + 14\mu^2 - 3$, the spectrum of $A_{ED_2}(G)$ is

$$Spec_{ED_2} = \begin{pmatrix} 3.1819 & 1.8019 & 0.4450 & -0.5936 & -1.2470 & -1.5884 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

and the energy of minimum equitable domination of ED_2 is $E_{ED_2} = 8.8578$ and $det_{ED_2}(G) = -3$. One can note that $det_{ED_2}(G) \neq 0$ and $E_{ED_2} > E_{ED_1}$. Also the energy of minimum equitable domination depends upon the minimum equitable domination set.

§2. Bounds for the Minimum Equitable Domination Energy of a Graph

We first need the following Lemma.

Lemma 2.1 *Let G be a graph with vertices $\{v_1, v_2, \ldots, v_n\}$ and let $A_{ED}(G)$ be the minimum equitable domination matrix of G. Let $\Phi_n(A_{ED}(G)) = det(\mu I_n - A_{ED}(G)) = c_0\mu^n + c_1\mu^{n-1} + c_2\mu^{n-2} + \cdots + c_n$ be the characteristic polynomial of $A_{ED}(G)$. Then*

(1) $c_0 = 1$;
(2) $c_1 = -|ED|$;
(3) $c_2 = \begin{pmatrix} |ED| \\ 2 \end{pmatrix} - m$.

Proof (1) $c_0 = 1$ follows directly from the definition $\Phi_n(G,\mu) = det(\mu I_n - A_{ED}(G))$, i.e $c_0 = 1$.

(2) $c_1 =$ sum of determinants of all 1×1 principal submatrices of $A_{ED}(G)$,

$$\text{i.e } c_1 = (-1)^1 \text{ trace of } A_{ED}(G) = -|ED|.$$

(3) $c_2 =$ sum of determinants of all 2×2 principal submatrices of $A_{ED}(G)$,

$$\begin{aligned} i.e\ c_2 &= (-1)^2 \sum_{i<j} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} = \sum_{i<j}(a_{ii}a_{jj} - a_{ij}a_{ji}) = \sum_{i<j} a_{ii}a_{jj} - \sum_{i<j} a_{ij}^2 \\ c_2 &= \begin{pmatrix} |ED| \\ 2 \end{pmatrix} - m. \qquad \square \end{aligned}$$

Lemma 2.2 *Let G be a connected graph and let $\mu_1, \mu_2, \cdots, \mu_n$ be the eigenvalues of minimum equitable dominating matrix $A_{ED}(G)$. Then*

$$\sum_{i=1}^{n} \mu_i = |ED|$$

and

$$\sum_{i=1}^{n} \mu_i^2 = 2m + |ED|.$$

Proof The sum of diagonal elements of $A_{ED}(G)$ is $\sum_{i=1}^{n} \mu_i = \text{trace}[A_{ED}(G)] = \sum_{i=1}^{n} a_{ii}$ $= |ED|$.

Similarly, the sum of squares of the eigenvalues of $A_{ED}(G)$ is trace of $[A_{ED}(G)]^2$,

$$\begin{aligned}\sum_{i=1}^{n} \mu_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\ a_{ji} \\ &= \sum_{i=1}^{n} (a_{ii})^2 + \sum_{i \neq j} a_{ij}\ a_{ji} \\ &= \sum_{i=1}^{n} (a_{ii})^2 + 2\sum_{i<j} (a_{ij})^2\end{aligned}$$

$$\sum_{i=1}^{n} \mu_i^2 = 2m + |ED|.$$ □

Theorem 2.3 *Let G_1 and G_2 be two graphs with n vertices and m_1, m_2 are number of edges in G_1 and G_2 respectively. Let $\mu_1, \mu_2, \cdots, \mu_n$ are eigenvalues of $A_{ED_1}(G_1)$ and $\mu'_1, \mu'_2, \cdots, \mu'_n$ are eigenvalues of $A_{ED_2}(G_2)$. Then*

$$\sum_{i=1}^{n} \mu_i\ \mu'_i \leq \sqrt{(2m_1 + |ED_1|)(2m_2 + |ED_2|)},$$

where $A_{ED_i}(G_i)$ is minimum equitable domination matrix of G_i $(i = 1, 2)$ and ED_1, ED_2 be minimum equitable domination sets of G_1 and G_2 respectively.

Proof Let $\mu_1, \mu_2, \cdots, \mu_n$ are eigenvalues of $A_{ED_1}(G_1)$ and $\mu'_1, \mu'_2, \cdots, \mu'_n$ are eigenvalues of $A_{ED_2}(G_2)$. Then by the Cauchy-Schwartz inequality, we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

If $a_i = \mu_i$, $b_i = \mu'_i$ then

$$\begin{aligned}\left(\sum_{i=1}^{n} \mu_i \mu'_i\right)^2 &\leq \left(\sum_{i=1}^{n} \mu_i^2\right)\left(\sum_{i=1}^{n} (\mu'_i)^2\right) \\ \left(\sum_{i=1}^{n} \mu_i \mu'_i\right)^2 &\leq (2m_1 + |ED_1|)\ (2m_2 + |ED_2|) \\ &\Rightarrow \sum_{i=1}^{n} \mu_i \mu'_i \leq \sqrt{(2m_1 + |ED_1|)\ (2m_2 + |ED_2|)}.\end{aligned}$$

Hence the theorem. □

Theorem 2.4 *Let G be a graph with n vertices, m edges. Let ED is the minimum equitable*

domination set. Then

$$\sqrt{(2m+|ED|)+n(n-1)\ |detA_{ED}(G)|^{2/n}}\ \leq\ E_{ED}(G)\ \leq\ \sqrt{n(2m+|ED|)}.$$

Proof This proof follows the ideas of McClelland's bounds [6] for graphs $E(G)$. For the upper bound, let $\mu_1,\mu_2,\cdots,\mu_n$ be the eigenvalues of minimum equitable dominating matrix $A_{ED}(G)$. Apply the Cauchy-Schwartz inequality to $(1,1,\cdots,1)$ and $(\mu_1,\mu_2,\cdots,\mu_n)$ is

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

If $a_i=1$, $b_i=|\mu_i|$ then

$$\begin{aligned}\left(\sum_{i=1}^{n}|\mu_i|\right)^2 &\leq \left(\sum_{i=1}^{n}1^2\right)\left(\sum_{i=1}^{n}|\mu_i|^2\right)\\ &\Rightarrow\ [E_{ED}(G)]^2 \leq n(2m+|ED|)\end{aligned}$$

from the above $\left(\sum_{i=1}^{n}\mu_i^2=2m+|ED|\right)$,

$$E_{ED}(G)\leq\sqrt{n(2m+|ED|)},$$

which is upper bound.

For the lower bound, by using arithmetic mean and geometric mean inequality, we have

$$\begin{aligned}\frac{1}{n(n-1)}\sum_{i\neq j}|\mu_i|\ |\mu_j| &\geq \left(\prod_{i\neq j}|\mu_i|\ |\mu_j|\right)^{\frac{1}{n(n-1)}}\\ \sum_{i\neq j}|\mu_i|\ |\mu_j| &\geq n(n-1)\left(\prod_{i=1}^{n}|\mu_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}}\end{aligned}$$

$$\sum_{i\neq j}|\mu_i|\ |\mu_j|\geq n(n-1)\left(\prod_{i=1}^{n}|\mu_i|\right)^{2/n}. \tag{3}$$

Consider

$$E_{ED}(G)]^2=\left[\sum_{i=1}^{n}|\mu_i|\right]^2=\sum_{i=1}^{n}|\mu_i|^2+\sum_{i\neq j}|\mu_i|\ |\mu_j|.$$

From (3) we have

$$[E_{ED}(G)]^2\geq\sum_{i=1}^{n}|\mu_i|^2+n(n-1)\left(\prod_{i=1}^{n}|\mu_i|\right)^{2/n},$$

$$[E_{ED}(G)]^2\geq\ (2m+|ED|)+n(n-1)\ |detA_{ED}(G)|^{2/n}$$

$$\Rightarrow [E_{ED}(G)] \ \geq \ \sqrt{(2m + |ED|) + n(n-1) \ |detA_{ED}(G)|^{2/n}},$$

which is lower bound. □

Theorem 2.5 *If the minimum equitable domination energy $E_{ED}(G)$ is a rational number, then $E_{ED}(G) \equiv |ED|(mod2)$, where ED is minimum equitable domination set.*

Proof Let $\mu_1, \mu_2, \cdots, \mu_n$ be the eigenvalues of minimum equitable domination matrix $A_{ED}(G)$. Then the trace of

$$A_{ED}(G) = \sum_{i=1}^{n} a_{ii} = |ED|.$$

Let $\mu_1, \mu_2, \cdots, \mu_r$ be positive and remaining eigenvalues are non-positive then,

$$\begin{aligned} E_{ED}(G) &= \sum_{i=1}^{n} |\mu_i| = (\mu_1 + \mu_2 + \cdots + \mu_r) - (\mu_{r+1} + \mu_{r+2} + \cdots + \mu_n) \\ &= 2(\mu_1 + \mu_2 + \cdots + \mu_r) - (\mu_1 + \mu_2 + \cdots + \mu_n) \\ &= 2(\mu_1 + \mu_2 + \ldots + \mu_r) - |ED| \\ \Rightarrow E_{ED}(G) &\equiv |ED|(mod2). \end{aligned}$$

Hence the theorem. □

§3. Minimum Equitable Domination Energy and Determinant of Certain Standard Graphs

Theorem 3.1 *For $n \geq 4$, the minimum equitable domination energy of star graph $S_{1,n-1}$ is $(n-2) + 2\sqrt{n-1}$.*

Proof Consider the star graph $S_{1,n-1}$ with vertex set $V = \{v_0, v_1, \cdots, v_{n-1}\}$. The minimum equitable domination set is $ED = \{v_0, v_1, \cdots, v_{n-1}\}$. Then minimum equitable domination matrix is

$$A_{ED}(S_{1,n-1}) = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 1 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \ldots & 1 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}_{n \times n}.$$

The characteristic polynomial of $A_{ED}(S_{1,n-1})$ is

$$\Phi_n(S_{1,n-1},\mu)=\begin{vmatrix}\mu-1 & -1 & -1 & \dots & -1 & -1\\ -1 & \mu-1 & 0 & \dots & 0 & 0\\ -1 & 0 & \mu-1 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ -1 & 0 & 0 & \dots & \mu-1 & 0\\ -1 & 0 & 0 & \dots & 0 & \mu-1\end{vmatrix}_{n\times n}$$

$$=-(-1)^{n+1}\begin{vmatrix}-1 & -1 & \dots & -1 & -1\\ \mu-1 & 0 & \dots & 0 & 0\\ 0 & \mu-1 & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots\mu-1 & 0 & 0\\ 0 & 0 & \dots & \mu-1 & 0\end{vmatrix}+(\mu-1)\begin{vmatrix}\mu-1 & -1 & \dots & -1 & -1\\ -1 & \mu-1 & \dots & 0 & 0\\ -1 & 0 & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ -1 & 0 & \dots & \mu-1 & 0\\ -1 & 0 & \dots & 0 & \mu-1\end{vmatrix}$$

$$\Phi_n(S_{1,n-1},\mu)=-(\mu-1)^{n-2}+(\mu-1)\Phi_{n-1}(S_{1,n-2},\mu). \tag{4}$$

Now change n to $n-1$ in (1), we get

$$\Phi_{n-1}(S_{1,n-2},\mu)=-(\mu-1)^{n-3}+(\mu-1)\Phi_{n-2}(S_{1,n-3},\mu). \tag{5}$$

Substitute (5) in (4),

$$\Phi_n(S_{1,n-1},\mu)=-2(\mu-1)^{n-2}+(\mu-1)^2\Phi_{n-2}(S_{1,n-3},\mu) \tag{6}$$

Continuing this process, we get

$$\begin{aligned}\Phi_n(S_{1,n-1},\mu) &= -(n-4)(\mu-1)^{n-2}+(\mu-1)^{n-4}\Phi_4(S_{1,3},\mu)\\ &= -(n-4)(\mu-1)^{n-2}+(\mu-1)^{n-4}[(\mu-1)^2(\mu^2-2\mu-2)]\\ \Phi_n(S_{1,n-1},\mu) &= (\mu-1)^{n-2}[\mu^2-2\mu-(n-2)].\end{aligned}$$

The spectrum of minimum domination energy of a graph is

$$Spec_{ED}(S_{1,\ n-1})=\begin{pmatrix}1+\sqrt{n-1} & 1 & 1-\sqrt{n-1}\\ 1 & n-2 & 1\end{pmatrix}.$$

The energy of minimum equitable domination of a graph is

$$E_{ED}(S_{1,\ n-1})=(n-2)+2\sqrt{n-1},\ n\geq 4. \qquad \square$$

Theorem 3.2 *Let $K_{s,t}$ be complete bipartite graph with $s+t$ vertices and energy of minimum equitable complete bipartite graph $E_{ED}(K_{s,t})$ is*

$$E_{ED}(K_{s,t})=(s+t-2)+2\sqrt{st}$$

if $|s-t|\geq 2,\ \ s,t\geq 2.$

Proof Let complete bipartite graph $K_{s,t}$ with $|s-t| \geq 2$ where $s,t \geq 2$ with vertex set $V = \{v_1, v_2, \cdots, v_s, u_1, u_2, \cdots, u_t\}$. The minimum equitable domination set is $ED = \{v_1, v_2, \cdots, v_s, u_1, u_2, \cdots, u_t\}$. Then

$$A_{ED}(K_{s,t}) = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 & 1 & 1 & \ldots & 1 & 1 \\ 0 & 1 & \ldots & 0 & 0 & 1 & 1 & \ldots & 1 & 1 \\ 0 & 0 & 1\ldots & 0 & 0 & 1 & 1 & \ldots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & 1 & 1 & \ldots & 1 & 1 \\ 0 & 0 & \ldots & 0 & 1 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 1 & \ldots & 1 & 1 & 1 & 0 & \ldots & 0 & 0 \\ 1 & 1 & \ldots & 1 & 1 & 0 & 1 & \ldots & 0 & 0 \\ 1 & 1 & \ldots & 1 & 1 & 0 & 0 & 1\ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \ldots & 1 & 1 & 0 & 0 & \ldots & 1 & 0 \\ 1 & 1 & \ldots & 1 & 1 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}_{(s+t)\times(s+t)}.$$

The characteristic polynomial of $A_{ED}(K_{s,t})$ is

$$\Phi_{s+t}(K_{s,t},\mu) = \begin{vmatrix} \mu-1 & 0 & \ldots & 0 & 0 & -1 & -1 & \ldots & -1 & -1 \\ 0 & \mu-1 & \ldots & 0 & 0 & -1 & -1 & \ldots & -1 & -1 \\ 0 & 0 & \ldots & 0 & 0 & -1 & -1 & \ldots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \mu-1 & 0 & -1 & -1 & \ldots & -1 & -1 \\ 0 & 0 & \ldots & 0 & \mu-1 & -1 & -1 & \ldots & -1 & -1 \\ -1 & -1 & \ldots & -1 & -1 & \mu-1 & 0 & \ldots & 0 & 0 \\ -1 & -1 & \ldots & -1 & -1 & 0 & \mu-1 & \ldots & 0 & 0 \\ -1 & -1 & \ldots & -1 & -1 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \ldots & -1 & -1 & 0 & 0 & \ldots & \mu-1 & 0 \\ -1 & -1 & \ldots & -1 & -1 & 0 & 0 & \ldots & 0 & \mu-1 \end{vmatrix}$$

$$= \begin{vmatrix} (\mu-1)I_s & -J_{s\times t}^T \\ -J_{t\times s} & (\mu-1)I_t \end{vmatrix},$$

where $J_{t\times s}$ is a matrix with all entries equal to one,

$$\begin{aligned}\Phi_{s+t}(K_{s,t},\mu) &= |(\mu-1)I_s|\ |(\mu-1)I_t-(-J)\frac{I_s}{\mu-1}(-J^T)| \\ &= (\mu-1)^{s-t}\ |(\mu-1)^2I_t-JJ^T| \\ &= (\mu-1)^{s-t}\ P_{JJ^T}[(\mu-1)^2] \\ &= (\mu-1)^{s-t}\ P_{sJ_t}[(\mu-1)^2],\end{aligned}$$

where P_{sJ_t} is the characteristic polynomial of the matrix sJ_t

$$\begin{aligned}\Phi_{s+t}(K_{s,t},\mu) &= (\mu-1)^{s-t}[(\mu-1)^2-st]\ [(\mu-1)^2]^{t-1} \\ &= (\mu-1)^{s-t}(\mu-1)^{2t-2}\ [\mu^2+1-2\mu-st] \\ \Phi_{s+t}(K_{s,t},\mu) &= (\mu-1)^{s+t-2}\ [\mu^2-2\mu-(st-1)]\end{aligned}$$

is the characteristic polynomial of minimum equitable domination matrix of $K_{s,t}$. The spectrum of minimum equitable domination matrix of $K_{s,t}$ is

$$Spec_{ED}(K_{s,\ t})=\begin{pmatrix}1+\sqrt{st} & 1 & 1-\sqrt{st}\\ 1 & s+t-2 & 1\end{pmatrix}.$$

The minimum equitable domination energy of a graph is

$$E_{ED}(K_{s,\ t})=(s+t-2)+2\sqrt{st}.\qquad \square$$

A *crown graph* S_n^0 for an integer $n\geq 3$ is the graph with vertex set $\{v_1,v_2,\ldots,v_n,u_1,u_2,\cdots,u_n\}$ and edge set $\{v_iu_i: 1\leq i,j\leq n,\ i\neq j\}$. Therefore S_n^0 coincides with the complete bipartite graph $K_{n,n}$ with horizontal edges removed [1].

Theorem 3.3 *For $n\geq 3$, the minimum equitable domination energy of the crown graph S_n^0 is equal to $2(n-2)+\sqrt{n^2-2n+5}+\sqrt{n^2+2n-3}$.*

Proof Consider crown graph S_n^0 with vertex set $V=\{v_1,v_2,\cdots,v_n,u_1,u_2,\cdots,u_n\}$. The minimum equitable domination set is $ED=\{v_1,u_1\}$.

Then the minimum equitable domination matrix of S_n^0 is same as the minimum domination matrix of S_n^0 by [7]. Therefore

$$E_{ED}(S_n^0)=2(n-2)+\sqrt{n^2-2n+5}+\sqrt{n^2+2n-3}.\qquad \square$$

Theorem 3.4 *For $n\geq 2$, the minimum equitable domination energy of complete graph K_n is $(n-2)+\sqrt{n^2-2n+5}$*

Proof Consider the complete graph K_n with vertex set $V=\{v_1,v_2,\cdots,v_n\}$. The minimum equitable domination set is $ED=\{v_1\}$. Then the minimum equitable domination matrix of K_n is

same as the minimum domination matrix of K_n by [7]. Therefore,

$$E_{ED}(K_n) = (n-2) + \sqrt{n^2-2n+5}. \qquad \square$$

Let us obtain the minimum equitable domination determinant of some graph G given by $det_{ED}(G) = \mu_1\mu_2\cdots\mu_n$, where $\mu_1,\ \mu_2,\cdots,\mu_n$ are eigenvalues of $A_{ED}(G)$.

Proposition 3.5 *Let $S_{1,n-1}$ $(n \geq 4)$, K_n $(n \geq 2)$ be the star and complete graphs with n vertices, respectively, S_n^0 $(n \geq 3)$, is crown graph with $2n$ vertices and $K_{s,t}$ $(|s-t| \geq 2)$ be the complete bipartite graph with $s+t$ vertices. Then*

(1) $det_{ED}(S_{1,n-1}) = -(n-2)$;
(2) $det_{ED}(K_n) = (-1)^{(n-1)}$;
(3) $det_{ED}(K_{s,t}) = (1-st)$;
(4) $det_{ED}(S_n^0) = (-1)^{n-1}(3-2n)$.

Proof w.k.t $det_{ED}(G) = \mu_1\ \mu_2\cdots\mu_n$, where $\mu_1,\mu_2,\cdots,\mu_n$ are eigenvalues of G.

Case 1. $det_{ED}(S_{1,n-1}) = (1+\sqrt{n-1})^1\ 1^{n-2}\ (1-\sqrt{n-1})^1 = -(n-2)$, where $n \geq 4$.

Case 2.

$$\begin{aligned} det_{ED}(K_n) &= \left(\frac{(n-1)+\sqrt{n^2-2n+5}}{2}\right)^1 (-1)^{n-2}\left(\frac{(n-1)-\sqrt{n^2-2n+5}}{2}\right)^1 \\ &= (-1)^{n-1},\ where\ \ n \geq 2. \end{aligned}$$

Case 3. $det_{ED}(K_{s,t}) = (1+\sqrt{st})^1\ (1)^{s+t-2}\ (1-\sqrt{st})^1 = (1-st)$, where $|s-t| \geq 2$.

Case 4.

$$\begin{aligned} det_{ED}(S_n^0) &= \left(\frac{(n-1)\pm\sqrt{n^2-2n+5}}{2}\right)^1 \left(\frac{(3-n)\pm\sqrt{n^2+2n-3}}{2}\right)^1 (1)^{n-2}(-1)^{n-2} \\ &= (-1)^{n-1}(3-2n), \end{aligned}$$

where $n \geq 3$. $\square$

Theorem 3.6 *If the graph G is non-singular (i.e no eigenvalues of $A_{ED}(G)$ is equal to zero) then $E_{ED}(G) \geq n$, (non-hypoenergetic).*

Proof. Let $\mu_1,\mu_2,\cdots,\mu_n$ are non-zero eigenvalues of $A_{ED}(G)$. Then inequality between the arithmetic and geometric mean, we have

$$\begin{aligned} \frac{dv}{dx}\frac{|\mu_1|+|\mu_2|+\cdots+|\mu_n|}{n} &\geq (|\mu_1||\mu_2|\cdots|\mu_n|)^{1/n} \\ \frac{1}{n}E_{ED}(G) &\geq (detA_{ED}(G))^{1/n}. \end{aligned}$$

The determinant of the $A_{ED}(G)$ matrix is necessary an integer. Because no eigenvalues is zero, $|detA_{ED}(G)| \geq 1$ then $|detA_{ED}(G)|^{1/n} \geq 1$. Therefore $E_{ED}(G) \geq n$. $\square$

Acknowledgement We acknowledge duly that the present research work is supported by the UGC-BSR Fellowship, Government of India, grant No.F.7-349/2011 and UGC-SAP, DRS-I, No. F.510/2/DRS /2011(SAP-I).

References

[1] C.Adiga, A.Bayad, I.Gutman and Shrikanth. A. S, The minimum covering energy of a graph, *Krag. J. Sci.*, 34 (2012), 39-56.

[2] R.B.Bapat, *Graphs and Matrices*, Hindustan Book Agency (2011).

[3] R.B.Bapat, S.Pati, Energy of a graph is never an odd integer, *Bull. Kerala Math. Assoc.*, 1 (2011), 129-132.

[4] D.M.Cvetkovic, M.Doob and H.Sachs, *Spectra of Graphs, Theory and Application*, Academic Press, New York, USA, 1980.

[5] I.Gutman, The energy of a graph, *Ber. Math-Statist. Sekt. Forschungsz. Graz*, 103 (1978), 1-22.

[6] B.J.McClelland, Properties of the latent roots of a matrix: The estimation of π-electron energies, *J. Chem. Phys.*, 54 (1971), 640-643.

[7] M.R.Rajesh Kanna, B.N.Dharmendra and G.Sridhara, The minimum dominating energy of a eraph, *Int. J. of Pure and Appl. Math.*, 85(4), 2013, 707-718.

[1] V.Swaminathan and K.M.Dharmalingam, Degree equitable domination on graphs, *Krag. J. of Math.*, 35 (1) (2011), 191-197.

[9] X. Li, Y.Shi and I. utman, *Graph Energy*, Springer New York, 2012.

Math.Combin.Book Ser. Vol.3(2015), 73-80

Some Results on Relaxed Mean Labeling

V.Maheswari[1], D.S.T.Ramesh[2] and V.Balaji[3]

1. Department of Mathematics, KCG College of Engineering and Technology, Chennai-600 095, India

2. Department of Mathematics, Margoschis College, Nazareth-628617, India

3. Department of Mathematics, Sacred Heart College, Tirupattur-635 601, India

E-mail: pulibala70@gmail.com

Abstract: In this paper, we investigate relaxed mean labeling of some standard graphs. We prove, any cycle is a relaxed mean graph; if n > 4, K_n is not a relaxed mean graph; $K_{2,n}$ is a relaxed mean graph for all n; any Triangular snake is a relaxed mean graph; any Quadrilateral snake is a relaxed mean graph; the graph P_n^2 is a relaxed mean graph; $L_n \Theta K_1$ is a relaxed mean graph. Also, we prove $K_n^c + 2K_2$ is a relaxed mean graph for all n; W_4 is a relaxed mean graph; $K_2 + mK_1$ is a relaxed mean graph for all m; if G_1 and G_2 are tree, then $G = G_1 \cup G_2$ is a relaxed mean graph; the planar grid $P_m \times P_n$ is a relaxed mean graph for $m \geqslant 2$, $n \geqslant 2$ and the prism $P_m \times C_n$ is a relaxed mean graph for $m \geqslant 2$ and for all $n \geqslant 3$.

Key Words: Smarandache relaxed k-mean graph, relaxed mean graph, cycle, path, star.

AMS(2010): 05C78.

§1. Introduction

All graphs in this paper are finite, simple an undirected. Terms not defined here are used in the sense of Harary [3]. In 1966, Rosa [5] introduced β-valuation of graph. Golomb subsequently called such a labeling graceful. In 1980, Graham and Sloane [2] introduced the harmonious labeling of a graph. Also, in 2003, Somasundaram and Ponraj [6] and [7] introduced the mean labeling of a graph. On similar lines, we define relaxed mean labeling. In [4], we proved any path is a relaxed mean graph and if $m = 5$, $K_{1,m}$ is not a relaxed mean graph. We proved the bistar $B_{m,n}$ is a relaxed mean graph if $|m-n| = 3$. Also, we proved that combs are relaxed mean graph and $C_3 \cup P_n$ is a relaxed mean graph for n $= 2$. In this paper, we prove any cycle is a relaxed mean graph; if n > 4, K_n is not a relaxed mean graph; $K_{2,n}$ is a relaxed mean graph for all n; any Triangular snake is a relaxed mean graph; any Quadrilateral snake is a relaxed mean graph; the graph P_n^2 is a relaxed mean graph; $L_n \Theta K_1$ is a relaxed mean graph and $K_n^c + 2K_2$ is a relaxed mean graph for all n. Also, we prove W_4 is a relaxed mean graph; $K_2 + mK_1$ is a relaxed mean graph for all m; If G_1 and G_2 are trees, then $G = G_1 \cup G_2$ is a relaxed mean graph; the planar grid $P_m \times P_n$ is a relaxed mean graph for m $\geqslant 2$, n $\geqslant 2$ and the prism $P_m \times C_n$ is a relaxed mean graph for m $\geqslant 2$ and for all n $\geqslant 3$. The condition for a graph to be relaxed mean is that $p = q + 1$ in [4].

[1]Received December 9, 2014, Accepted August 18, 2015.

§2. Main Results

Definition 2.1 *A graph $G = (V, E)$ with p vertices and q edges is said to be a Smarandache relaxed k-mean graph if there exists a function f from the vertex set of G to $\{0, 1, 2, 3, \cdots, q+1\}$ such that in the induced map $f*$ from the edge set of G to $\{1, 2, 3, \cdots, q\}$ defined by*

$$f*(e = uv) = \begin{cases} \dfrac{f(u)+f(v)}{2} & if\ f(u)+f(v)\ is\ even \\ \dfrac{f(u)+f(v)+1}{2} & if\ \mathrm{f(u)}+f(v)\ is\ odd,\ then \end{cases}$$

the resulting edge labels are distinct. Furthermore, such a graph is called a Smarandache relaxed k-mean graph if we replace 2 by k and $f*(uv)$ by

$$\left\lfloor \frac{f(u)+f(v)}{k} \right\rfloor.$$

Theorem 2.2 *Any cycle is a relaxed mean graph.*

Proof The proof is divided into two cases following.

Case 1. Let n be odd. Let C_n be a cycle $u_1 u_2 \cdots u_n u_1$. Define f : $V(C_n) \to \{0, 1, 2, \cdots, q = n\}$ and $q + 1 = n + 1$ by $f(u_1) = 0$; $f(u_n) = n + 1$; $f(u_i) = i - 1$ for $2 \leqslant i \leqslant \frac{n-1}{2}$ and $f(u_j) = j$ for $\frac{n+1}{2} \leqslant j \leqslant n - 1$.

Case 2. Let n be even. Let C_n be a cycle $u_1 u_2 \cdots {}_n u_1$. Define f : $V(C_n) \to \{0, 1, 2, \cdots, q = n\}$ and $q + 1 = n + 1$ by $f(u_1) = 0$; $f(u_n) = n + 1$; $f(u_i) = i - 1$ for $2 \leqslant i \leqslant \frac{n}{2}$ and $f(u_j) = j$ for $\frac{n}{2} + 1 \leqslant j \leqslant n - 1$.

Therefore, the set of labels of the edges of C_n is $\{1, 2, \ldots, n\}$. Hence C_n is a relaxed mean graph. □

Theorem 2.3 *If $n > 4$, K_n is not a relaxed mean graph.*

Proof Suppose $n > 4$, K_n is a relaxed mean graph. To get the edge label $q + 1 = \frac{n(n-1)}{2} + 1$, we must have $q + 1$ and $q - 2$ as the vertex labels. Let u and v be the vertices whose vertex labels are $q + 1$ and $q - 2$ respectively.

To get the edge label 1 we must have 0 and 1 as the vertex label (or) 0 and 2 as the vertex label. In either case 0 must be a label of some vertex. Let w be the vertex whose vertex label is 0.

If $q + 1$ is even, the edges uw and vw get the same label $\frac{q+2}{2}$ which should not happen. If $q + 1$ is odd and 0,1 are the vertex labels with labels w_1 having vertex label 1, then the edges uw and uw_1 get the same label $\frac{q+2}{2}$; if q is odd and 0, 2 are the vertex labels with w_1 having vertex label 2, then the edges uw and vw_1 get the same label $\frac{q+2}{2}$ which again should not happen. Hence K_n is not a relaxed mean graph for $n > 4$. □

Theorem 2.4 *$K_{2,n}$ is a relaxed mean graph for all n.*

Proof Let (V_1, V_2) be the bipartition of $K_{2,n}$ with $V_1 = \{u, v\}$, $V_2 = \{u_1, u_2, \cdots, u_n\}$. Define f: $V(K_{2,n}) \to \{0, 1, 2, \ldots, q = 2n\}$ and $q + 1 = 2n + 1$ by $f(u) = 1$; $f(v) = 2n + 1$; $f(u_1) = 0$ and $f(u_{i+1}) = 2i$ for $1 \leqslant i \leqslant n - 1$.

The corresponding edge labels are as follows:

The label of the edge uu_1 is 1. The label of the edge uu_{i+1} is $i + 1$ for $1 \leqslant i \leqslant n - 1$. The label of the edge vu_{i+1} is $n + i + 1$ for $1 \leqslant i \leqslant n - 1$. The label of the edge vu_1 is $n + 1$. Hence $K_{2,n}$ is a relaxed mean graph for all n. □

Definition 2.5 *A triangular snake is obtained from a path $v_1v_2 \cdots v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $1 \leqslant i \leqslant n - 1$. That is, every edge of a path is replaced by a triangle C_3.*

Theorem 2.6 *Any triangular snake is a relaxed mean graph.*

Proof Let T_n be a triangular snake. Define $f : V(T_n) \to \{0, 1, 2, \ldots, q = 3n - 3\}$ and $q + 1 = 3n - 2$ by $f(v_i) = 3i - 3$ for $1 \leqslant i \leqslant n - 1$ and $f(v_n) = 3n - 2$; $f(w_i) = 3i - 1$ for $\leqslant i \leqslant n - 2$ and $f(v_{n-1}) = 3n - 5$.

The corresponding edge labels are as follows:

The labels of the edge $v_{i-1}v_i$ is $3i - 4$ for $2 \leqslant i \leqslant n - 1$. The labels of the edge $v_{n-1}v_n$ is $3n - 4$. The labels of the edge w_iv_i is $3i - 2$ for $1 \leqslant i \leqslant n - 2$. The labels of the edge $w_{n-1}v_{n-1}$ is $3n - 5$. The labels of the edge $w_{i-1}v_i$ is $3i - 3$ for $2 \leqslant i \leqslant n - 1$. The labels of the edge $w_{n-1}v_n$ is $3n - 3$. Hence T_n is a relaxed mean graph. □

Definition 2.7 *A quadrilateral snake is obtained from a path $u_1u_2 \ldots u_n$ by joining u_i, u_{i+1} to new vertices v_i, w_i respectively and joining v_i and w_i. That is, every edge of a path is replaced by a cycle C_4.*

Theorem 2.8 *Any quadrilateral snake is a relaxed mean graph.*

Proof Let Q_n denote a quadrilateral snake. Define $f : V(Q_n) \to \{0, 1, 2, \ldots, q = 4n - 4\}$ and $q + 1 = 4n - 3$ by $f(u_i) = 4i - 4$ for $1 \leqslant i \leqslant n - 1$ and $f(u_n) = 4n - 3$. $f(v_i) = 4i - 2$ for $1 \leqslant i \leqslant n - 2$ and $f(v_{n-1}) = 4n - 7$. $f(w_i) = 4i - 1$ for $1 \leqslant i \leqslant n - 2$ and $f(w_{n-1}) = 4n - 6$.

The corresponding edge labels are as follows:

The labels of the edge $u_{i-1}u_i$ is $4i - 6$ for $2 \leqslant i \leqslant n-1$ and $u_{n-1}u_n$ is $4n-5$. The labels of the edge u_iv_i is $4i - 3$ for $1 \leqslant i \leqslant n - 2$ and $u_{n-1}v_{n-1}$ is $4n - 7$. The labels of the edge $u_{i+1}w_i$ is $4i$ for $1 \leqslant i \leqslant n - 2$ and u_nw_{n-1} is $4n - 4$. The labels of the edge v_iw_i is $4i-1$ for $1 \leqslant i \leqslant n - 2$ and $v_{n-1}w_{n-1}$ is $4n - 6$. Hence Q_n is a relaxed mean graph. □

Definition 2.9 *The square G^2 of a graph G has $V(G^2) = V(G)$ with u,v is adjacent in G^2 whenever $d(u, v) \leqslant 2$ in G. The powers G^3, G^4, ... of G are similarly defined.*

Theorem 2.10 *The graph P_n^2 is a relaxed mean graph.*

Proof Let $u_1u_2 \ldots u_n$ be the path P_n. Clearly, P_n^2 has n vertices and $2n-3$ edges. Define $f : V(P_n^2) \to \{0, 1, 2, \ldots, q = 2n - 3\}$ and $q + 1 = 2n - 2$ by $f(u_i) = 2i - 2$ for $1 \leqslant i \leqslant n - 2$; $f(u_{n-1}) = 2n - 5$ and $f(u_n) = 2n - 2$.

The corresponding edge labels are as follows:

The labels of the edge u_iu_{i+1} is $2i - 1$ for $1 \leqslant i \leqslant n - 2$ and $u_{n-1}u_n$ is $2n - 3$. The labels of the edge u_iu_{i+2} is $2i$ for $1 \leqslant i \leqslant n - 2$. Hence P_n^2 is a relaxed mean graph. □

Definition 2.11 *The graph $C_3^{(t)}$ denotes that the one point union of t copies of cycle C_n. The graph $C_3^{(t)}$ is called a friendship graph or Dutch t-windmill.*

The graph $C_3^{(t)}$ is a relaxed mean graph. For instance, $C_3^{(4)}$ is shown in Fig.1.

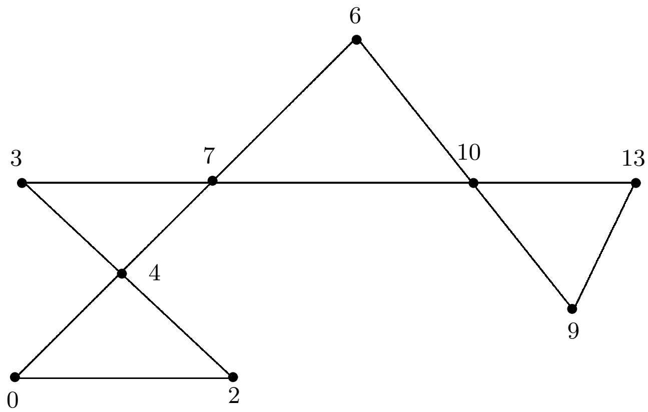


Fig.1

Theorem 2.12 *Let C_n be the cycle $u_1u_2\ldots u_nu_1$. Let G be a graph with V(G) = V(C_n) $\cup$ $\{w_i$: $1 \leqslant i \leqslant n\}$ and E(G) = E(C_n) $\cup$ $\{u_iw_i$, $u_{i+1}w_i$: $1 \leqslant i \leqslant n\}$. Then G is a relaxed mean graph.*

Proof The proof is divided into two cases following.

Case 1. n is odd.

Define $f : V(G) \to \{0, 1, 2, \cdots, q = 3n\}$ and $q+1 = 3n+1$ by $f(u_i) = 3i-3$ for $1 \leqslant i \leqslant (n-1)/2$; $f(w_i) = 3i-1$ for $1 \leqslant i \leqslant (n-1)/2$; $f(u_{(n+1)/2}) = (3n-1)/2$; $f(u_{(n+3)/2}) = (3n+9)/2$; $f(u_{(n+3)/2+i}) = (3n+9)/2 + 3i + 1$ for $1 \leqslant i \leqslant (n-3)/2$; $f(w_{(n+1)/2}) = (3n+7)/2$; $f(w_{(n+3)/2}) = (3n+5)/2$ and $f(w_{(n+3)/2+i}) = (3n+7)/2 + 3i - 1$ for $1 \leqslant i \leqslant (n-3)/2$. Clearly, f is a relaxed mean labeling of G.

Case 2. n is even and $n \geqslant 8$.

Define f : V(G) $\to$ {0, 1, 2, ..., q = 3n} and q + 1 = 3n + 1 by $f(u_1) = 3$; $f(u_i) = 3i - 4$ for $2 \leqslant i \leqslant n/2$; $f(u_{(n/2)+1}) = (3n/2) + 1$; $f(u_{(n/2)+i}) = (3n/2) - 2 + 3i$ for $2 \leqslant i \leqslant (n - 4)/2$; $f(u_{n-1}) = 3n - 3$; $f(u_n) = 3n + 1$; $f(w_1) = 0$; $f(w_2) = 7$; $f(w_i) = 3i + 1$ for $3 \leqslant i \leqslant (n - 2)/2$; $f(w_{(n/2)}) = (3n - 2)/2$; $f(w_{(n/2)+1}) = (3n + 12)/2$; $f(w_{(n/2)+i+1}) = (3n + 12)/2 + 3i$ for $1 \leqslant i \leqslant (n - 8)/2$; $f(w_{n-2}) = 3n - 4$; $f(w_{n-1}) = 3n - 5$ and $f(w_n) = 3n - 2$.

Clearly, f is a relaxed mean labeling of G. Hence G is a relaxed mean graph. □

Theorem 2.13 *Let C_n be the cycle $u_1u_2\ldots u_nu_1$. Let G be a graph with V(G) = V(C_n) and E(G) = E(C_n) $\cup$ $\{u_1u_3\}$. Then G is a relaxed mean graph.*

Proof The proof is divided into two cases.

Case 1. n is odd.

Define f : V(C_n) $\to$ {0, 1, 2, ..., q = n + 1} and q + 1 = n + 2 by $f(u_1) = 0$ and $f(u_n) = n + 2$. Also, $f(u_i) = i$ for i = 2, 3 ; $f(u_j) = j + 1$ for $\frac{n+1}{2} \leqslant j \leqslant n-1$ and $f(u_k) = k$ for $k \neq i, j$.

Case 2. n is even.

Define f : $V(C_n) \to \{0, 1, 2, \ldots, q = n + 1\}$ and $q + 1 = n + 2$ by $f(u_1) = 0$ and $f(u_n) = n + 2$.
Also, $f(u_i) = i$ for $i = 2, 3$; $f(u_j) = j + 1$ for $\frac{n}{2} \leqslant j \leqslant n - 1$and $f(u_k) = k$ for $k \neq i, j$.
Clearly, f is a relaxed mean labeling of G. □

Theorem 2.14 *Let C_n be the cycle $u_1u_2 \ldots u_nu_1$. Let G be a graph with $V(G) = V(C_n)$ and $E(G) = E(C_n) \cup \{u_3u_6\}$. Then G is a relaxed mean graph.*

Proof The proof is divided into two cases.

Case 1. *n* is odd.

Define f : $V(C_n) \to \{0, 1, 2, \ldots, q = n + 1\}$ and $q + 1 = n + 2$ by $f(u_1) = 0$; $f(u_2) = 2$; $f(u_3) = 1$ and $f(u_n) = n + 2$.
Also, $f(u_i) = i + 1$ for $\frac{n+1}{2} \leqslant i \leqslant n - 1$ and $f(u_j) = j + 1$ for $i \neq j$.

Case 2. *n* is even.

Define f : $V(C_n) \to \{0, 1, 2, \ldots, q = n + 1\}$ and $q + 1 = n + 2$ by $f(u_1) = 0$; $f(u_2) = 2$; $f(u_3) = 1$ and $f(u_n) = n + 2$.
Also, $f(u_i) = i + 1$ for $\frac{n}{2} \leqslant i \leqslant n - 1$ and $f(u_j) = j + 1$ for $i \neq j$.
Clearly, f is a relaxed mean labeling of G. Hence G is a relaxed mean graph. □

Definition 2.15 *The graph* $L_n = P_n \times P_1$ *is called the ladder.*

We proceed to corona with ladder.

Theorem 2.16 *$L_n \ominus K_1$ is a relaxed mean graph.*

Proof Let $V(L_n) = \{a_i, b_i : 1 \leqslant i \leqslant n\}$ and $E(L_n) = \{a_ib_i : 1 \leqslant i \leqslant n - 1\} \cup \{a_ia_{i+1} : 1 \leqslant i \leqslant n - 1\} \cup \{b_ib_{i+1} : 1 \leqslant i \leqslant n - 1\}$.

Let c_i be the pendent vertex adjacent to a_i and let d_i be the pendent vertex adjacent to b_i. Define f: $V(L_n \ominus K_1) \to \{0, 1, 2, \ldots, q = 5n - 2\}$ and $q + 1 = 5n - 1$ by $f(a_i) = 5i - 4$ for $1 \leqslant i \leqslant n$; $f(b_i) = 5i - 3$ for $1 \leqslant i \leqslant n$; $f(c_i) = 5i - 5$ for $1 \leqslant i \leqslant n$; $f(d_i) = 5i - 2$ for $1 \leqslant i \leqslant n - 1$ and $f(d_n) = 5n - 1$.

The corresponding edge labels are as follows:

The labels of the edge c_ia_i is $5i - 4$ for $1 \leqslant i \leqslant n$. The labels of the edge a_ib_i is $5i - 3$ for $1 \leqslant i \leqslant n$. The labels of the edge b_id_i is $5i - 2$ for $1 \leqslant i \leqslant n$. The labels of the edge a_ia_{i+1} is $5i - 1$ for $1 \leqslant i \leqslant n - 1$. The labels of the edge b_ib_{i+1} is $5i$ for $1 \leqslant i \leqslant n - 1$.

Clearly, f is a relaxed mean labeling of G. Hence $L_n \ominus K_1$is a relaxed mean graph. □

Definition 2.17 *The graph* $K_n^c + 2K_2$ *is the join of complement of the complete graph on n vertices and two disjoint copies of* K_2. *First we prove that* $K_n^c + 2K_2$ *is a relaxed mean graph.*

Theorem 2.18 $K_n^c + 2K_2$ *is a relaxed mean graph for all* n.

Proof Let $V(K_n) = \{u_1, u_2, ..., u_n\}$, $V(2K_2) = \{u, v, w, z\}$ and
$E(2K_2) = \{uv, wz\}$.

Define f : $V(K_n^c + 2K_2) \to \{0, 1, 2, \ldots, q = 4n + 2\}$ and $q + 1 = 4n + 3$ by $f(u) = 2$, $f(v) = 0$, $f(w) = 4n + 3$, $f(z) = 4n$ and $f(u_i) = 4i - 1$ for $1 \leqslant i \leqslant n$.

The corresponding edge labels are as follows:

The label of the edge uv is 1. The label of the edge wz is $4n+2$. The label of the edge uu_i is $2i+1$ for $1 \leqslant i \leqslant n$. The label of the edge vu_i is $2i$ for $1 \leqslant i \leqslant n$. The label of the edge wu_i is $2n+2i+1$ for $1 \leqslant i \leqslant n$. The label of the edge zu_i is $2n + 2i$ for $1 \leqslant i \leqslant n$.

Hence $K_n^c + 2K_2$ is a relaxed mean graph for all n. □

The wheel W_n is the join of the graphs C_n and K_1. Next we investigate the relaxed mean labeling of the wheel $W_n = C_n + K_1$. The wheel $W_3 = K_4$ is a relaxed mean graph. We investigate W_n for any n, we take the case $n = 4$.

Theorem 2.19 it W_4 is a relaxed mean graph.

Proof Suppose W_4 is a relaxed mean graph with labeling f. Let $W_4 = C_4 + K_1$, where C_4 is the cycle $u_1u_2u_3u_4u_1$ and $V(K_1) = \{u\}$. To get the edge label 1 either 0 and 1 or 0 and 2 are the vertex labels of adjacent vertices. To get the edge label 8, 9 and 6 must be the vertex label of adjacent vertices. Let 0 and 2 are the vertex labels of adjacent vertices.

Then $f(u) = 6$; $f(u_i) = 0$; $f(u_{i+1}) = 2$; $f(u_{i+2}) = 9$ and $f(u_{i+3}) = 4$ for some i, $1 \leqslant i \leqslant 4$. Therefore, the induced edge labels are distinct.

Clearly, f is a relaxed mean labeling of G. Hence W_4 is a relaxed mean graph. □

Definition 2.20 $K_2 + mK_1$ *is the join of the graph* K_2 *and* m *disjoint copies of* K_1. *Some authors call this graph a Book with triangular pages. We now investigate the relaxed mean labeling of* $K_2 + mK_1$.

Theorem 2.21 $K_2 + mK_1$ *is a relaxed mean graph for all* m.

Proof Let u, v be the vertices of K_2 and $u_1, u_2, \ldots, u_m$ be the remaining vertices of $K_2 + mK_1$. Define $f : V(K_2 + mK_1) \to \{0, 1, 2, \ldots, q = 2m + 1\}$ and $q + 1 = 2m + 2$ by $f(u) = 0$, $f(v) = 2m + 2$, $f(u_i) = 2i$ for $1 \leqslant i \leqslant m-1$; $f(u_m) = 2m-1$. The label of the edge uu_i is i for $1 \leqslant i \leqslant m-1$. The label of the edge uv is $m + 1$. The label of the edge vu_i is $m + 1 + i$ for $1 \leqslant i \leqslant m-1$. The label of the edge uu_m is m. The label of the edge vu_m is $2m + 1$. Therefore the induced edge labels are distinct.

Clearly, f is a relaxed mean labeling of G. Hence $K_2 + mK_1$ is a relaxed mean graph. □

Theorem 2.22 *If* G_1 *and* G_2 *are trees, then* $G = G_1 \cup G_2$ *is a relaxed mean graph.*

Proof Let $G_1 = (p_1, q_1)$, $G_2 = (p_2, q_2)$ be the given trees and let G be a (p, q) graph.

Therefore, $p = p_1 + p_2$ and $q = q_1 + q_2$. Since G_1 and G_2 are trees, $q_1 = p_1 - 1$ and $q_2 = p_2 - 1$.

Now, $q + 1 = q_1 + q_2 + 1 = p_1 - 1 + p_2 - 1 + 1 = p_1 + p_2 - 1 = p - 1$. Whence, $G = G_1 \cup G_2$ is a relaxed mean graph. □

Theorem 2.23 *The planar grid* $P_m \times P_n$ *is a relaxed mean graph for* $m \geqslant 2$ *and* $n \geqslant 2$.

Proof Let $V(P_m \times P_n) = \{a_{ij} : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$ and

$$E(P_m \times P_n) = \left\{a_{i(j-1)}a_{ij} : 1 \leqslant i \leqslant m,\ 2 \leqslant j \leqslant n\right\} \cup \left\{a_{(i-1)j}a_{ij} : 2 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\right\}.$$

Define $f : V(P_m \times P_n) \to \{0, 1, 2, \ldots, q = 2mn - (m+n)\}$ and $q + 1 = 2mn - (m + n - 1)$ by

$f(a_{1j}) = j-1$, $1 \leqslant j \leqslant n$ and

$$f(a_{ij}) = \begin{cases} f\left(a_{(i-1)n}\right) + (n-1) + j, & 2 \leqslant i \leqslant m, 1 \leqslant j \leqslant n. \\ f\left(a_{(i-1)n}\right) + (n-1) + j + 1 & if\ m\ \ and\ n\ \ are\ maximum \end{cases}$$

The label of the edge $a_{ij}a_{i(j+1)}$ is $(i-1)(2n-1)+j$ *for* $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n-1$.

The label of the edge $a_{ij}a_{(i+1)j}$ is $(n-1)+(i-1)(2n-1)+j$ *for* $1 \leqslant i \leqslant m-1$, $1 \leqslant j \leqslant n$.

Clearly, f is a relaxed mean labeling of G Hence $P_m \times P_n$ is a relaxed mean graph for $m \geqslant 2$ and $n \geqslant 2$. □

Theorem 2.24 *The prism $P_m \times C_n$ is a relaxed mean graph for $m \geqslant 2$ and for all $n \geqslant 3$.*

Proof Let $V(P_m \times C_n) = \{a_{ij} : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$ and

$$E(P_m \times C_n) = \left\{a_{i\,(j-1)}\ a_{i\,j} : 1 \leqslant i \leqslant m,\ 2 \leqslant j \leqslant n\right\} \bigcup \left\{a_{(i-1)\,j}\ a_{ij} : 2 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\right\} \bigcup \{a_{i1}\ a_{in} : 1 \leqslant i \leqslant m\}.$$

Take

$$n = \begin{cases} 2r & if\ n\ is\ even \\ 2r+1 & if\ n\ is\ odd \end{cases}$$

Define $f : V(P_m \times C_n) \rightarrow \{0, 1, 2, ..., q+1 = (2mn-n)+1\}$ by $f(a_{1j}) = 2j$ for $1 \leqslant j \leqslant r$ and $f(a_{1n}) = 0$, Also,

$$f(a_{1\ r+j}) = \begin{cases} n-2j+2 & \text{if } n \text{ is odd for } 1 \leqslant j \leqslant r \\ n-2j+1 & \text{if } n \text{ is even for } 1 \leqslant j \leqslant r-1 \end{cases}$$

$$f(a_{2j}) = 2n + 2(j-1) \ \text{ for } 1 \leqslant j \leqslant r+1$$

$$f(a_{2\ r+1+j}) = \begin{cases} 3n-2j+2 & \text{if } n \text{ is odd for } 1 \leqslant j \leqslant r \\ 3n-2j+1 & \text{if } n \text{ is even for } 1 \leqslant j \leqslant r-1 \end{cases}$$

$$f\left(a_{(2i-1)\ j}\right) = f(a_{1j}) + 4n(i-1) \ \text{ for } 2 \leqslant i \leqslant \frac{m-1}{2}, 1 \leqslant j \leqslant n\ ;$$

Also, $f(a_{(2i-1)1}) = f(a_{11}) + 5m$ for $i = \dfrac{m+1}{2}$;

$f(a_{(2i-1)2}) = f(a_{12}) + 5m + 2$ for $i = \dfrac{m+1}{2}$ and f $(a_{(2i-1)3})$ = f (a_{13}) + 5m + 1 for $i = \dfrac{m+1}{2}$.
$f\left(a_{(2i-2)\ j}\right) = f(a_{2j}) + 4n(i-2)$ *for* $3 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$.

Clearly, f is a relaxed mean labeling of $P_m \times C_n$ for $m \geqslant 2$, $n \geqslant 3$. Hence G is a relaxed mean graph.

References

[1] J.A.Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, 16(2010), # DS6.

[2] R.L.Graham and N.J.A. , On additive bases and harmonious graphs, *SIAM J. Alg. Discrete Math.*, 1 (1980) 382 – 404.

[3] F.Harary, *Graph Theory*, Addison-Wesley, Reading Mars., 1972.

[4] V.Maheswari, D.S.T.Ramesh and V.Balaji, Relaxed mean labeling, *International Journal of Mathematics Research*, Volume 4, 3 (2012), 217-224.

[5] Rosa, On certain valuation of the vertices of a graph, *Theory of Graphs* (International Symposium, Rome, July 1966), Gorden and Breach, N.Y and Dunod Paris (1967) - 349 – 355.

[6] S.Somasundaram and R.Ponraj, Mean labeling of graphs, *National Academic Science Letters*, 26(2003), 210 – 213.

[7] S.Somasundaram and R.Ponraj, Some results on mean graphs, *Pure and Applied Mathematika Sciences*, 58(2003), 29 – 35.

Math.Combin.Book Ser. Vol.3(2015), 81-87

Split Geodetic Number of a Line Graph

Venkanagouda M Goudar and Ashalatha K.S

Sri Gauthama Research Centre

Department of Mathematics, Sri Siddhartha Institute of Technology, Tumkur, Karnataka, India

Venkatesha

Department of Mathematics, Kuvempu University Shankarghatta, Shimoga, Karnataka, India

E-mail: vmgouda@gmail.com, ashu7kslatha@gmail.com, vensprem@gmail.com

Abstract: A set $S \subseteq V[L(G)]$ is a split geodetic set of $L(G)$, if S is a geodetic set and $\langle V - S \rangle$ is disconnected. The split geodetic number of a line graph $L(G)$, denoted by $g_s[L(G)]$ is the minimum cardinality of a split geodetic set of $L(G)$. In this paper we obtain the split geodetic number of line graph of any graph. Also obtain many bounds on split geodetic number in terms of elements of G and covering number of G. We also investigate the relationship between split geodetic number and geodetic number.

Key Words: Label Cartesian product, distance, edge covering number, line graph, Smarandache k-split geodetic set, split geodetic number, vertex covering number.

AMS(2010): 05C05, 05C12.

§1. Introduction

In this paper we follow the notations of [3]. As usual $n = |V|$ and $m = |E|$ denote the number of vertices and edges of a graph G respectively. The graphs considered here have at least one component which is not complete or at least two non trivial components.

For any graph $G(V, E)$, the line graph $L(G)$ whose vertices correspond to the edges of G and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. It is well known that this distance is a metric on the vertex set $V(G)$. For a vertex v of G, the eccentricity $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is radius, rad G, and the maximum eccentricity is the diameter, diam G. A $u - v$ path of length d(u, v) is called a $u - v$ geodesic. We define $I[u, v]$ to the set (interval) of all vertices lying on some $u - v$ geodesic of G and for a nonempty subset S of $V(G)$, $I[S] = \bigcup_{u,v \in S} I[u, v]$.

A set S of vertices of G is called a geodetic set in G if $I[S] = V(G)$, and a geodetic set of minimum cardinality is a minimum geodetic set. The cardinality of a minimum geodetic set in G is called the geodetic number of G, and we denote it by $g(G)$.

Split geodetic number of a graph was studied by in [5]. A *Smarandache k-split geodetic set S* of a graph $G = (V, E)$ is such a split geodetic set that the induced subgraph $\langle V - S \rangle$ is k-connected. Particularly, if $k = 0$, such a split geodetic set is called a *split geodetic set* S of a graph G. The split geodetic number $g_s(G)$ of G is the minimum cardinality of a split geodetic set. Geodetic number of a

[1]Received September 22, 2014, Accepted August 2, 2015.

line graph was studied by in [4]. Geodetic number of a line graph $L(G)$ of G is a set $S^{'}$ of vertices of $L(G) = H$ is called the geodetic set in H if $I(S^{'}) = V(H)$ and a geodetic set of minimum cardinality is the geodetic number of $L(G)$ and is denoted by $g[L(G)]$. Now we define split geodetic number of a line graph. A set $S^{'}$ of vertices of $L(G) = H$ is called the split geodetic set in H if the induced subgraph $V(H) - S^{'}$ is disconnected and a split geodetic set of minimum cardinality is the split geodetic number of $L(G)$ and is denoted by $g_s[L(G)]$.

A vertex v is an extreme vertex in a graph G, if the subgraph induced by its neighbors is complete. A vertex cover in a graph G is a set of vertices that covers all edges of G. The minimum number of vertices in a vertex cover of G is the vertex covering number $\alpha_0(G)$ of G. An edge cover of a graph G without isolated vertices is a set of edges of G that covers all the vertices of G. The edge covering number $\alpha_1(G)$ of a graph G is the minimum cardinality of an edge cover of G.

For terminologies and notations not mentioned here, we follow references [2] and [3].

§2. Preliminary Notes

We need results following for proving results in this paper.

Theorem 2.1([1]) *Every geodetic set of a graph contains its extreme vertices.*

Theorem 2.2([5]) *For cycle C_n of order $n > 3$,*

$$g_s(C_n) = \begin{cases} 2 & \text{if n is even} \\ 3 & \text{if n is odd.} \end{cases}$$

Theorem 2.3([1]) *Let G be a connected graph of order at least 3. If G contains a minimum geodetic set S with a vertex x such that every vertex of G lies on some $x - w$ geodesic in G for some $w \in S$, then $g(G) = g(G \times K_2)$.*

Proposition 2.4 *For any graph G, $g(G) \leq g_s(G)$.*

Proposition 2.5 *For any tree T of order n and number of cut vertices c_i then the number of end edges is $n - c_i$.*

§3. Main Results

Theorem 3.1 *For any tree T with k end edges and c_i be the number of cut vertices, having more than three internal vertices, $g_s[L(T)] = n - c_i + 1$.*

Proof Let S be the set of all extreme vertices of a line graph $L(T)$ of a tree T. Let v_i be a cut vertex in $V - S$ and $S^{'} = S \cup \{v_i\}$. By Theorem 2.1, $g_s[L(T)] \geq |S^{'}|$. On the other hand, for an internal vertex v of $L(T)$, there exists x, y of $L(T)$ such that v lies on a unique $x - y$ geodesic in $L(T)$. The corresponding end edges of T are the extreme vertices of $L(T)$ and the induced subgraph $V - S^{'}$ is disconnected. Thus $g_s[L(T)] \leq |S^{'}|$. Also, every split geodetic set S_1 of $L(T)$ must contain $S^{'}$ which is the unique minimum split geodetic set. Thus $|S^{'}| = |S_1| = k + 1$. By Proposition 2.5, $|S_1| = n - c_i + 1$. Hence, $g_s[L(T)] = n - c_i + 1$. □

Corollary 3.2 *For any path P_n, $n \geq 6$, $g_s[L(P_n)] = 3$.*

Proof Clearly, the set of two end vertices of a path P_n is its unique geodetic set. From Theorem 3.1, the results follows. □

Proposition 3.3 *Line graph of a cycle is again a cycle of same order.*

Theorem 3.4 *For cycle C_n of order $n > 3$,*

$$g_s[L(C_n)] = \begin{cases} 2 & \text{if n is even} \\ 3 & \text{if n is odd.} \end{cases}$$

Proof The result follows from Proposition 3.3 and Theorem 2.2. □

Theorem 3.5 *For the wheel $W_n = K_1 + C_{n-1}$ $(n \geq 6)$,*

$$g_s[L(W_n)] = \begin{cases} \frac{n}{2} + 2 & \text{if n is even} \\ \frac{n+1}{2} + 2 & \text{if n is odd.} \end{cases}$$

Proof Let $W_n = K_1 + C_{n-1} (n \geq 6)$ and let $V(W_n) = \{x, v_1, v_2, \cdots, v_{n-1}\}$, where $deg(x) = n-1 > 3$ and $deg(v_i) = 3$ for each $i \in \{1, 2, \cdots, n-1\}$. Now $U = \{u_1, u_2, \cdots, u_j\}$ are the vertices of $L(W_n)$ formed from edges of C_{n-1}, i.e., $U \subseteq V[L(W_n)]$ and $Y = \{y_1, y_2, \cdots, y_j\}$ are the vertices of $L(W_n)$ formed from internal edges of W_n. Thus, $Y \subseteq V[L(W_n)]$. We consider the following cases.

Case 1. n is even.

Let $H \subseteq U$. Now $S = H \cup \{y_j\}$ forms a minimum geodetic set of $L(W_n)$. Let $P = \{p_1, p_2, \cdots, p_i\}$ be the vertices of $V[L(W_n)] - S$. Clearly, $S \cup \{p_l, p_k\}$ forms a minimum split geodetic set of $L(W_n)$ and $|S \cup \{p_l, p_k\}| = \frac{n}{2} + 2$. Therefore, $g_s[L(W_n)] = \frac{n}{2} + 2$.

Case 2. n is odd.

Let $H \subseteq U$, now $S = H \cup \{y_j, y_{j-1}\}$ forms a minimum geodetic set of $L(W_n)$. Let $P = \{p_1, p_2, \cdots, p_i\}$ be the vertices of $V[L(W_n)] - S$. Now $S \cup \{p_l, p_k\}$ forms a minimum split geodetic set of $L(W_n)$. Clearly, $|S \cup \{p_l, p_k\}| = \frac{n+1}{2} + 2$. Therefore, $g_s[L(W_n)] = \frac{n+1}{2} + 2$. □

As an immediate consequence of the above theorem we have the following.

Corollary 3.6 *For the wheel $W_n = K_1 + C_{n-1}$ $(n \geq 6)$,*

$$g_s[L(W_n)] = \begin{cases} \frac{\Delta + \delta}{2} & \text{if n is even} \\ \frac{\Delta + \delta + 1}{2} & \text{if n is odd.} \end{cases}$$

Proof Minimum degree(δ) of $L(W_n)$ is equal to 4 and maximum degree(Δ) of $L(W_n)$ is equal to n. i,e number of vertices in W_n.

Case 1. n is even.

We have known from Case 1 of Theorem 3.5 that

$$\begin{aligned} g_s[L(W_n)] &= \frac{n}{2}+2 \\ g_s[L(W_n)] &= \frac{n+4}{2} \\ g_s[L(W_n)] &= \frac{\Delta+\delta+1}{2} \end{aligned}$$

Case 2. n is odd.

We have known from Case 2 of Theorem 3.5 that

$$\begin{aligned} g_s[L(W_n)] &= \frac{n+1}{2}+2 \\ g_s[L(W_n)] &= \frac{n+4+1}{2} \\ g_s[L(W_n)] &= \frac{\Delta+\delta+1}{2}. \end{aligned}$$

□

Theorem 3.7 *For the wheel $W_n = K_1 + C_{n-1}$ $(n \geq 6)$, $g_s[L(W_n)] + g[L(W_n)] \leq m$.*

Proof Let $U = \{u_1, u_2, \cdots, u_j\} \subseteq V[L(W_n)]$ be the set of vertices formed from edges of C_{n-1} and $Y = \{y_1, y_2, \cdots, y_j\} \subseteq V[L(W_n)$ be the set of vertices formed from internal edges of W_n. Consider $S = H \cup \{y_j\}$, where $H \subset U$ forms a minimum geodetic set of $L(W_n)$. Furthermore, if $P = \{p_1, p_2, \cdots, p_i\}$ is the set of vertices of $V[L(W_n)] - S$, then $S' = S \cup \{p_l, p_m\}$ forms a minimum split geodetic set of $L(W_n)$. Notice that $V[L(G)] = E(G) = m$. It follows that $|S'| \cup |S| \leq m$. Thus, $g_s[L(W_n)] + g[L(W_n)] \leq m$.□

Theorem 3.8 *For a tree T with more than three internal vertices, $g_s[L(T)] \geq m - \alpha_1 + 1$, where α_1 is the edge covering number.*

Proof Suppose $S = \{e_1, e_2, \cdots, e_k\}$ to be the set of all end edges in T. Then $S \cup J$, where $J \subseteq E(T) - S$ is the minimal set of edges which covers all the vertices of T such that $|S \cup J| = \alpha_1(T)$. Without loss of generality, let $I = \{u_1, u_2, \cdots, u_n\} \subseteq V[L(T)]$ be the set of vertices in $L(T)$ formed by the end edges in T. Suppose $H = \{u_1, u_2, \cdots, u_i\} \subseteq V[L(T)] - I$. Then $I \cup \{u_i\}$ forms a minimum split geodetic set of $L(T)$, where each $u_i \in H$ with $deg \geq 2$. Clearly, it follows that $|I \cup \{u_i\}| \geq |E(T)| - |S \cup J| + 1$. Therefore, $g_s[L(T)] \geq m - \alpha_1(T) + 1$. □

Theorem 3.9 *If every non end vertex of a tree T with more than three internal vertex is adjacent to at least one end vertex, then $g_s[L(T)] \geq n - k$, where k is the number of end vertices in T.*

Proof Let $S' = \{v_1, v_2, \cdots, v_k\}$ be the set of all end vertices in T with $|S'| = k$. Without loss of generality, let every end edge of T be the extreme vertices of $L(T)$. Suppose $L(T)$ does not contain any end vertex. Then $S = I \cup \{u_j\}$, where $I = \{u_1, u_2, \cdots, u_i\} \subseteq V[L(T)]$ and $u_j \in V[L(T)] - I$ with $deg \geq 2$ forms a minimum split geodetic set of $L(T)$. Furthermore, if $L(T)$ contain at least one end vertex v_i, then the set $S \cup \{v_i\}$ forms a minimum split geodetic set of $L(T)$. Therefore, we obtain $|S \cup \{v_i\}| \geq n - |S'|$. Clearly it follows that $g_s[L(T)] \geq n - k$. □

Theorem 3.10 *For any connected graph G of order n, $g_s(G) + g_s[L(G)] \leq 2n$.*

Proof Let $S = \{v_1, v_2, \cdots, v_n\} \subseteq V(G)$ be the minimum split geodetic set of G. Now without loss of generality, if $F = \{u_1, u_2, \cdots, u_k\}$ is the set of all end vertices in $L(G)$, then $F \cup H$, where $H \subseteq V[L(G)] - F$ forms a minimum split geodetic set of $L(G)$. Since each vertex in $L(G)$ corresponds

to two adjacent vertices of G, it follows that $|S| \cup |F \cup H| \leq 2n$. Therefore $g_s(G) + g_s[L(G)] \leq 2n$. $\square$

Theorem 3.11 *Let G be a connected graph of order n with diameter $d > 4$. Then $g_s[L(G)] \leq n-d+2$.*

Proof Let u and v be vertices of $L(G)$ for which $d(u,v) = d$ and let $u = v_0, v_1, \cdots, v_d = v$ be the $u-v$ path of length d. Now let $S = V[L(G)] - \{v_1, v_2, \cdots, v_{d-1}\}$. Then $I(S) = V[L(G)]$, $V[L(G)] - (S \cup \{v_2\})$ is disconnected and $g_s[L(G)] \leq |S| = n-d+2$. $\square$

Theorem 3.12 *For any integers $r, s \geq 2$, $g_s[L(K_{r,s})] \leq rs$.*

Proof Notice that the diameter of $L(K_{r,s})$ is 2 and the number of vertices in $L(K_{r,s})$ is rs. By Theorem 3.11, $g_s[L(G)] \leq n-d+2$. Now we have $g_s[L(K_{r,s})] \leq rs-2+2$. Therefore, $g_s[L(K_{r,s})] \leq rs$. $\square$

Theorem 3.13 *For any integer $n \geq 4$, $g_s[L(K_n)] \leq \dfrac{n(n-1)}{2}$.*

Proof Let $n \geq 4$ be the vertices of the given graph K_n with diameter d. Since diameter of $L(K_n)$ is 2 and the number of vertices in $L(K_n)$ is $\dfrac{n(n-1)}{2}$. By Theorem 3.11, $g_s[L(G)] \leq n-d+2$. We have

$$g_s[L(K_n)] \leq \frac{n(n-1)}{2} - 2 + 2 \Rightarrow g_s[L(K_n)] \leq \frac{n(n-1)}{2}. \qquad \square$$

Theorem 3.14 *For any cycle C_n with $n \equiv 0(\mathrm{mod}2)$, $g_s[L(C_n)] = \dfrac{n}{\alpha_0(C_n)}$, where α_0 is the vertex covering number.*

Proof Let $n > 3$ be number of vertices which is even and let α_0 be the vertex covering number of C_n. By Theorem 3.4, $g_s[L(C_n)] = 2$. Also, for even cycle, the vertex covering number $\alpha_0(C_n) = \dfrac{n}{2}$. Hence $g_s[L(C_n)] = 2 = \dfrac{n}{n/2} = \dfrac{n}{\alpha_0(C_n)}$. $\square$

Theorem 3.15 *For any cycle C_n with $n \equiv 1(\mathrm{mod}2)$, $g_s[L(C_n)] = \dfrac{n+1}{\alpha_0(C_n)} + 1$, where α_0 is the vertex covering number.*

Proof Let $n > 3$ be the number of vertices which is odd and let α_0 be the vertex covering number of C_n. By Theorem 3.4, $g_s[L(C_n)] = 3$. Also, for odd cycle, vertex covering number $\alpha_0(C_n) = \dfrac{n+1}{2}$. Hence $g_s[L(C_n)] = 2+1 = \dfrac{n+1}{\alpha_0(C_n)} + 1$. $\square$

§4. Adding an End Edge

For an edge $e = (u,v)$ of a graph G with $deg(u) = 1$ and $deg(v) > 1$, we call e an end-edge and u an end-vertex.

Theorem 4.1 *Let G' be the graph obtained by adding k end edges $\{(u,v_1), (u,v_2), \cdots, (u,v_k)\}$ to a cycle $C_n = G$ of order $n > 3$, with $u \in G$ and $\{v_1, v_2, \cdots, v_k\} \notin G$. Then $g_s[L(G')] = k+2$.*

Proof Let $\{e_1, e_2, \cdots, e_n, e_1\}$ be edges on a cycle of order n and let G' be the graph obtained from $G = C_n$ by adding end edges (u, v_i), $i = 1, 2, \cdots, k$ such that $u \in G$ but $v_i \notin G$.

Case 1. n is even.

By definition, $L(G^{'})$ has $\langle K_{k+2}\rangle$ as an induced subgraph. Also the edges (u, v_i), $i = 1, 2, \cdots, k$ becomes vertices of $L(G^{'})$ and it belongs to some geodetic set of $L(G^{'})$. Hence $\{e_1, e_2, \cdots, e_k, e_l, e_m\}$ are the vertices of $L(G^{'})$, where e_l, e_m are the edges incident on the antipodal vertex of u in $G^{'}$ and these vertices belongs to some geodetic set of $L(G^{'})$. $L(G^{'}) = C_n \cup K_{k+2}$. Let$S = \{e_1, e_2, \cdots, e_k, e_l, e_m\}$ be the geodetic set. Suppose $P = \{e_1, e_2, \cdots, e_k\}$ is the set of vertices of $L(G^{'})$ such that $|P| < |S|$. Then, P is not a geodetic set of $L(G^{'})$. Clearly, S is the minimum geodetic set. Since $V - S$ is disconnected S is the minimum split geodetic set. Therefore $g_s[L(C_{2n})] = k + 2$.

Case 2. n is odd.

By definition, $L(G^{'})$ has $\langle K_{k+2}\rangle$ as an induced subgraph, also the edges $(u, v_i) = \{e_1, e_2, \cdots, e_k\}$ becomes vertices of $L(G^{'})$. Let $e_l = (a, b) \in G$ such that $d(u, a) = d(u, b)$ in the graph $L(G^{'})$. and let $S = \{e_1, e_2, \cdots, e_k, e_l\}$ be the geodetic set. Now $S^{'} = S \cup \{e_m\}$ is a split geodetic set, where e_m is the vertex from $V - S$ with $deg \geq 2$. It is clear that $S^{'}$ is the minimum split geodetic set. Therefore $g_s[L(C_{2n+1})] = k + 2$. □

Theorem 4.2 *Let $G^{'}$ be the graph obtained by adding end edge (u_i, v_j), $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, k$ to each vertex of $G = C_n$ of order $n > 3$ such that $u_i \in G$, $v_j \notin G$. Then $g_s[L(G^{'})] = k + 2$.*

Proof Let $\{e_1, e_2, \cdots, e_n, e_1\}$ be edges on a cycle $G = C_n$ and let $G^{'}$ be the graph obtained by adding end edge (u_i, v_j), $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, k$ to each vertex of G such that $u_i \in G$ but $v_j \notin G$. Clearly, k be the number of end vertices of $G^{'}$. By definition, $L(G^{'})$ have n copies of K_3 as an induced subgraph. The edges $(u_i, v_j) = e_j$ for all j becomes k vertices of $L(G^{'})$ and those lies on geodetic set of $L(G^{'})$. They form the extreme vertices of $L(G^{'})$. By Theorem 2.1 $S = \{e_1, e_2, \cdots, e_k\}$ forms a geodetic set. Now consider any two vertices $\{e_l, e_m\} \in V - S$ which are not adjacent. $S^{'} = \{e_1, e_2, \cdots, e_k, e_l, e_m\}$ forms a split geodetic set of $L(G^{'})$. Suppose $P = \{e_1, e_2, \cdots, e_k, e_l\}$ is the set of vertices of $L(G^{'})$ such that $|P| < |S^{'}|$. Then, $V - P$ is connected. Hence it is clear that $S^{'}$ is the minimum split geodetic set of $L(G^{'})$. There fore $g_s[L(G^{'})] = k + 2$. □

§5. Cartesian Product

The Cartesian product of the graphs H_1 and H_2, written as $H_1 \times H_2$, is the graph with vertex set $V(H_1) \times V(H_2)$, two vertices u_1, u_2 and v_1, v_2 being adjacent in $H_1 \times H_2$ if and only if either $u_1 = v_1$ and $(u_2, v_2) \in E(H_2)$, or $u_2 = v_2$ and $(u_1, v_1) \in E(H_1)$.

Theorem 5.1 *For any path P_n of order n,*

$$g_s[L(K_2 \times P_n)] = \begin{cases} 2 & \text{for } n = 2 \\ 3 & \text{for } n = 3 \\ 4 & \text{for } n > 3. \end{cases}$$

Proof Let $K_2 \times P_n$ be formed from two copies of G_1 and G_2 of P_n. Now, $L(K_2 \times P_n)$ formed from two copies of $G_1^{'}, G_2^{'}$ of $L(P_n)$. And let $U = \{u_1, u_2, \cdots, u_{n-1}\} \in V(G_1^{'})$, $W = \{w_1, w_2, \cdots, w_{n-1}\} \in V(G_2^{'})$. We have the following cases.

Case 1. If $n = 2$, by the definition $L(K_2 \times P_2) = K_2 \times P_2$. By Theorem 2.3,

$$g_s[L(K_2 \times P_2)] = g[L(K_2 \times P_2)] = g(P_2) = 2.$$

Case 2. If $n = 3$, $L(K_2 \times P_3)$ is formed from two copies of P_2. Clearly, $g_s[L(K_2 \times P_3)] = 3$.

Case 3. If $n > 3$, let S be the split geodetic set of $L(K_2 \times P_n)$. We claim that S contains two elements (end vertices) from each set $\{u_1, u_{n-1}, w_1, w_{n-1}\}$ and $V - S$ is disconnected. Since $I(S) = V[L(K_2 \times P_n)]$, it follows that $g_s[L(K_2 \times P_n)] \leq 4$. It remains to show that if $S^{'}$ is a three element subset of $V[L(K_2 \times P_n)]$, then $I(S^{'}) \neq V[L(K_2 \times P_n)]$. First assume that $S^{'}$ is a subset U or W, say the former. Then $I(S^{'}) = S^{'} \cup W \neq V$. Therefore, we may take that $S^{'} \cap U = \{u_i, u_j\}$ and $S^{'} \cap W = \{w_k\}$. Then

$$I(S^{'}) = \{u_i, u_j\} \cup W \neq V[L(K_2 \times P_n)].$$ □

References

[1] G.Chartrand, F.Harary and P.Zhang, On the geodetic number of a graph, *Networks*, 39(2002), 1-6.

[2] G.Chartrand and P.Zhang, *Introduction to Graph Theory*, Tata McGraw Hill Pub.Co. Ltd., 2006.

[3] F.Harary, *Graph Theory*, Addison-Wesely, Reading, MA, 1969.

[4] Venkanagouda M.Goudar, K.S.Ashalatha, Venkatesha and M.H.Muddebihal, On the geodetic number of line graph, *Int.J.Contemp.Math.Sciences*, Vol.7, No.46(2012), 2289-2295.

[5] Venkanagouda M.Goudar, K.S.Ashalatha and Venkatesha, Split geodetic number of a graph, *Advances and Applications in Discrete Mathematics*, Vol.13, No.1(2014), 9-22.

Math.Combin.Book Ser. Vol.3(2015), 88-98

Skolem Difference Odd Mean Labeling For Some Simple Graphs

R.Vasuki, J.Venkateswari and G.Pooranam

Department of Mathematics, Dr.Sivanthi Aditanar College of Engineering

Tiruchendur- 628 215, Tamilnadu, India

E-mail: vasukisehar@gmail.com, revathi198715@gmail.com, dpooranamg@gmail.com

Abstract: A graph G with p vertices and q edges is said to have skolem difference odd mean labeling if there exists an injective function $f : V(G) \to \{1, 2, 3, \cdots, 4q-1\}$ such that the induced map $f^* : E(G) \to \{1, 3, 5, \cdots, 2q-1\}$ defined by

$$f^*(uv) = \begin{cases} \dfrac{|f(u)-f(v)|}{2} & \text{if } |f(u)-f(v)| \text{ is even} \\ \dfrac{|f(u)-f(v)|+1}{2} & \text{if } |f(u)-f(v)| \text{ is odd} \end{cases}$$

is a bijection. A graph that admits skolem difference odd mean labeling is called a skolem difference odd mean graph. Here we investigate skolem difference odd mean behaviour of some standard graphs.

Key Words: Labeling, skolem difference odd mean graph, Smarandache k-mean graph.

AMS(2010): 05C78.

§1. Introduction

Throughout this paper, by a graph we mean a finite, undirected and simple graph. Let $G(V, E)$ be a graph with p vertices and q edges. For notations and terminology we follow [1].

Path on n vertices is denoted by P_n and a cycle on n vertices is denoted by C_n. A graph $G = (V, E)$ is called bipartite if $V = V_1 \cup V_2$ with $\phi = V_1 \cap V_2$, and every edge of G is of the form $\{u, v\}$ with $u \in V_1$ and $v \in V_2$. If each vertex in V_1 is joined with every vertex in V_2, we have a complete bipartite graph. In this case $|V_1| = m$ and $|V_2| = n$, the graph is denoted by $K_{m,n}$. The complete bipartite graph $K_{1,n}$ is called a star graph and it is denoted by S_m. The bistar $B_{m,n}$ is the graph obtained from K_2 by identifying the center vertices of $K_{1,m}$ and $K_{1,n}$ at the end vertices of K_2 respectively. $B_{m,m}$ is often denoted by $B(m)$.

A quadrilateral snake Q_n is obtained from a path $u_1, u_2, \cdots, u_{n+1}$ by joining u_i and u_{i+1} to new vertices v_i and w_i respectively and joining v_i and w_i, $1 \le i \le n$, that is, every edge of a path is replaced by a cycle C_4. The corona of a graph G on p vertices $v_1, v_2, \cdots, v_p$ is the graph obtained from G by adding p new vertices $u_1, u_2, \cdots, u_p$ and the new edges u_iv_i for $1 \le i \le p$. The corona of G is denoted by $G \odot K_1$. The graph $P_n \odot K_1$ is called a comb. Let G_1 and G_2 be any two graphs with p_1 and p_2 vertices respectively. Then the cartesian product $G_1 \times G_2$ has p_1p_2 vertices which are $\{(u, v)/u \in G_1, v \in G_2\}$. The edge set of $G_1 \times G_2$ is obtained as follows: (u_1, v_1) and (u_2, v_2) are adjacent in $G_1 \times G_2$ if either

[1]Received March 24, 2014, Accepted May 15, 2015.

$u_1 = u_2$ and v_1 and v_2 are adjacent in G_2 or u_1 and u_2 are adjacent in G_1 and $v_1 = v_2$. The product $P_m \times P_n$ is called a planar grid and $P_n \times P_2$ is called a ladder, denoted by L_n. The graph $P_2 \times P_2 \times P_2$ is called a cube and is denoted by Q_3. A dragon is a graph formed by joining the end vertex of a path to a vertex of the cycle.

The concept of mean labeling was introduced and studied by S. Somasundaram and R. Ponraj [5]. Some new families of mean graphs are studied by S.K. Vaidya et al. [10]. Further some more results on mean graphs are discussed in [4,6,7]. A graph G is said to be a mean graph if there exists an injective function f from $V(G)$ to $\{0, 1, 2, \cdots, q\}$ such that the induced map f^* from $E(G)$ to $\{1, 2, 3, \cdots, q\}$ defined by $f^*(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ is a bijection. Furthermore, if f^* is defined by $f^*(uv) = \left\lceil \frac{f(u)+f(v)}{k} \right\rceil$ for an integer $k \geq 2$ hold with previous properties, then G is called a *Smarandache k-mean graph*.

In [2], K. Manickam and M. Marudai introduced odd mean labeling of a graph. A graph G is said to be odd mean if there exists an injective function f from $V(G)$ to $\{0, 1, 2, 3, \cdots, 2q-1\}$ such that the induced map f^* from $E(G)$ to $\{1, 3, 5, \cdots, 2q-1\}$ defined by $f^*(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ is a bijection. Some more results on odd mean graphs are discussed in [8,9].

The concept of skolem difference mean labeling was introduced and studied by K. Murugan and A. Subramanian [3]. A graph $G = (V, E)$ with p vertices and q edges is said to have skolem difference mean labeling if it is possible to label the vertices $x \in V$ with distinct elements $f(x)$ from $1, 2, 3 \cdots, p+q$ in such a way that for each edge $e = uv$, let $f^*(e) = \left\lceil \frac{|f(u)-f(v)|}{2} \right\rceil$ and the resulting labels of the edges are distinct and are from $1, 2, 3, \cdots, q$. A graph that admits a skolem difference mean labeling is called a skolem difference mean graph. It motivates us to define a new concept called skolem difference odd mean labeling.

A graph with p vertices and q edges is said to have a skolem difference odd mean labeling if there exists an injective function $f : V(G) \to \{1, 2, 3, \cdots, 4q-1\}$ such that the induced map $f^* : E(G) \to \{1, 3, 5, \cdots, 2q-1\}$ defined by $f^*(uv) = \left\lceil \frac{|f(u)-f(v)|}{2} \right\rceil$ is a bijection. A graph that admits a skolem difference odd mean labeling is called a skolem difference odd mean graph.

For example, a skolem difference odd mean labeling of cube Q_3 shown in Fig.1.

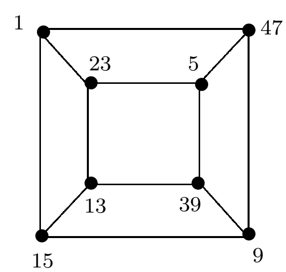


Fig.1

In this paper, we prove that the path P_n, the cycle C_n for $n \geq 4$, $K_{m,n} (m \geq 1, n \geq 1)$, the bistar $B_{m,n}$ for $m \geq 1, n \geq 1$, the quadrilateral snake Q_n, the ladder $L_n, L_n \odot K_1$ and $K_{1,n} \odot K_1$ for $n \geq 1$ are skolem difference odd mean graphs.

§2. Skolem Difference Odd Mean Graphs

Theorem 2.1 *Any path is a skolem difference odd mean graph.*

Proof Let $u_1, u_2, \ldots, u_n$ be the vertices of the path P_n. Define $f : V(P_n) \to \{1, 2, 3, \cdots, 4q-1 =$

$4n-5\}$ as follows:

$$f(u_{2i-1}) = 4i-3, \qquad 1 \le i \le \left\lceil \frac{n}{2} \right\rceil$$
$$f(u_{2i}) = 4n-4i-1, \qquad 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$$

The label of the edge u_iu_{i+1} is $2n-2i-1$, $1 \le i \le n-1$. Hence, P_n is a skolem difference odd mean graph. $\square$

For example, a skolem difference odd mean labeling of P_8 and P_{11} are shown in Fig.2.

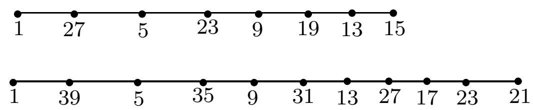


Fig.2

Theorem 2.2 *Cycle C_n is a skolem difference odd mean graph for $n \ge 4$.*

Proof Let $u_1, u_2, \cdots, u_n$ be the vertices of the cycle C_n. Define $f : V(C_n) \to \{1, 2, 3, \cdots, 4q-1 = 4n-1\}$ as follows:

Case 1. $n \equiv 0(mod\ 4)$

$$f(u_i) = \begin{cases} 2i-1, & 1 \le i \le n \text{ and } i \text{ is odd,} \\ 4n-2i+3, & 1 \le i \le \frac{n}{2} \text{ and } i \text{ is even,} \\ 4n-2i-1, & \frac{n+4}{2} \le i \le n \text{ and } i \text{ is even.} \end{cases}$$

For the vertex labeling f, the induced edge labeling f^* is given as follows:

$$f^*(u_iu_{i+1}) = \begin{cases} 2n-2i+1, & 1 \le i \le \frac{n}{2} \\ 2n-2i-1, & \frac{n+2}{2} \le i \le n-1 \end{cases}$$

and $$f^*(u_nu_1) = n-1.$$

Case 2. $n \equiv 1(mod\ 4)$

$$f(u_i) = \begin{cases} 2i-1, & 1 \le i \le n-2 \text{ and } i \text{ is odd} \\ 4n-2i+3, & 1 \le i \le \frac{n-1}{2} \text{ and } i \text{ is even} \\ 4n-2i-1, & \frac{n+3}{2} \le i \le n-1 \text{ and } i \text{ is even} \\ 2n, & i = n. \end{cases}$$

For the vertex labeling f, the induced edge labeling f^* is obtained as follows:

$$f^*(u_iu_{i+1}) = \begin{cases} 2n-2i+1, & 1 \le i \le \frac{n-1}{2} \\ 2n-2i-1, & \frac{n+1}{2} \le i \le n-1 \end{cases}$$

and $$f^*(u_nu_1) = n.$$

Case 3. $n \equiv 2(mod\ 4)$

$$f(u_i)=\begin{cases} 2i-1, & 1\le i\le \frac{n}{2} \text{ and } i \text{ is odd}\\ 2i+3, & \frac{n+4}{2}\le i\le n-1 \text{ and } i \text{ is odd},\\ 4n-2i+3, & 1\le i\le n-2 \text{ and } i \text{ is even},\\ 2n-1, & i=n.\end{cases}$$

For the vertex labeling f, the induced edge labeling f^* is given as follows:

$$f^*(u_iu_{i+1})=\begin{cases} 2n-2i+1, & 1\le i\le \frac{n}{2}\\ 2n-2i-1, & \frac{n+2}{2}\le i\le n-1\end{cases}$$

and

$$f^*(u_nu_1)=n-1.$$

Case 4. $n\equiv 3(mod\ 4)$

$$f(u_i)=\begin{cases} 2i-1, & 1\le i\le n-4 \text{ and } i \text{ is odd}\\ 2n-4, & i=n-2\\ 4n-2, & i=n\\ 4n-2i-1, & 1\le i\le \frac{n-3}{2} \text{ and } i \text{ is even}\\ 4n-2i-5, & \frac{n+1}{2}\le i\le n-3 \text{ and } i \text{ is even}\\ 2n-2, & i=n-1.\end{cases}$$

For the vertex labeling f, the induced edge labeling f^* is obtained as follows:

$$f^*(u_iu_{i+1})=\begin{cases} 2n-2i-1, & 1\le i\le \frac{n-3}{2}\\ 2n-2i-3, & \frac{n-1}{2}\le i\le n-2\\ n, & i=n-1.\end{cases}$$

and

$$f^*(u_nu_1)=2n-1.$$

Then, f is a skolem difference odd mean labeling. Thus, C_n for $n\ge 4$ is a skolem difference odd mean graph. □

For example, a skolem difference odd mean labeling of C_{12} and C_{11} are shown in Fig.3.

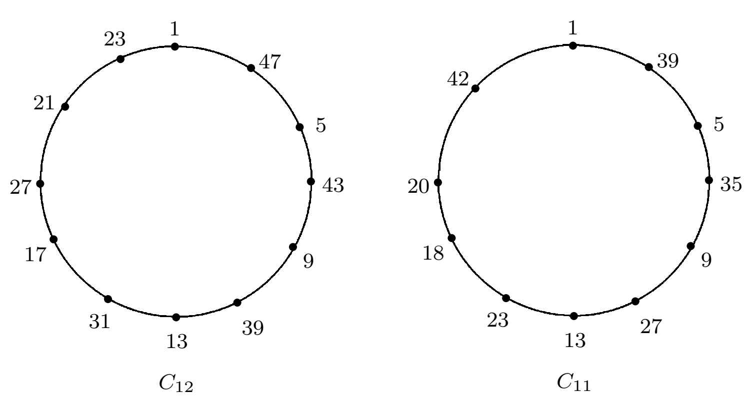


Fig.3

Theorem 2.3 *Every complete bipartite graph $K_{m,n}(m \geq 1, n \geq 1)$ is a skolem difference odd mean graph.*

Proof Let $V = V_1 \cup V_2$ where $V_1 = \{u_1, u_2, \cdots, u_m\}$ and $V_2 = \{v_1, v_2, \cdots, v_n\}$. The graph $K_{m,n}$ has $m+n$ vertices and mn edges. Define $f : V(K_{m,n}) \rightarrow \{1, 2, 3, \cdots, 4q-1 = 4mn-1\}$ as follows:

$$f(u_i) = 4i - 3, \qquad 1 \leq i \leq m$$
$$f(v_j) = 4mn - 4m(j-1) - 1, \qquad 1 \leq j \leq n$$

For the vertex labeling f, the induced edge label f^* is obtained as $f^*(u_iv_j) = 2mn - 2i + 1 - 2m(j-1), 1 \leq i \leq m, 1 \leq j \leq n$. Then, f gives a skolem difference odd mean labeling. Hence, $K_{m,n}$ is a skolem difference odd mean graph for all $m \geq 1, n \geq 1$. □

For example, a skolem difference odd mean labeling of $K_{4,5}$ is shown in Fig.4.

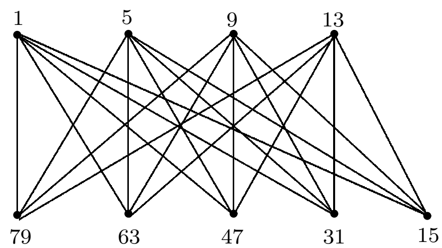


Fig.4

Corollary 2.4 *By taking $m = 1$, in the proof of the above theorem, we get a star graph $K_{1,n}$ and it is a skolem difference odd mean graph.*

Theorem 2.5 *The bistar $B_{m,n}$ is a skolem difference odd mean graph for $m \geq 1, n \geq 1$.*

Proof Let $V(K_2) = \{u, v\}$ and $u_i(1 \leq i \leq m)$, $v_j(1 \leq j \leq n)$ be the vertices adjacent to u and v respectively. Define $f : V(B_{m,n}) \rightarrow \{1, 2, \cdots, 4q-1 = 4(m+n)+3\}$ by

$$f(u) = 1,$$
$$f(v) = 4(m+n) + 3,$$
$$f(u_i) = 4i - 1, \quad 1 \leq i \leq m,$$
$$f(v_j) = 4j + 1, \quad 1 \leq j \leq n.$$

The induced edge labels are given as follows:

$$f^*(uv) = 2m + 2n + 1,$$
$$f^*(uu_i) = 2i - 1, \quad 1 \leq i \leq m$$
$$f^*(vv_j) = 2m + 2n - 2j + 1, \quad 1 \leq j \leq n.$$

Then, f is a skolem difference odd mean labeling and hence $B_{m,n}$ is a skolem difference odd mean graph for all $m \geq 1, n \geq 1$. □

For example, a skolem difference odd mean labeling of $B_{4,7}$ is shown in Fig.5.

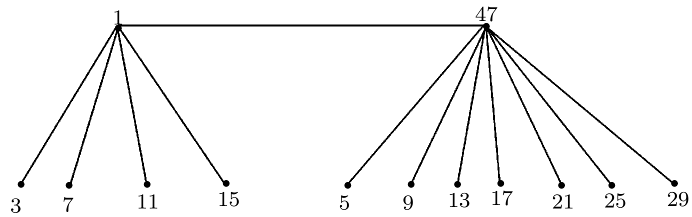


Fig.5

Theorem 2.6 *A quadrilateral snake is a skolem difference odd mean graph.*

Proof Let Q_n denote the quadrilateral snake obtained from $u_1, u_2, \ldots, u_{n+1}$ by joining u_i, u_{i+1} to new vertices v_i, w_i respectively and joining v_i and $w_i, 1 \leq i \leq n$. The graph Q_n has $3n+1$ vertices and $4n$ edges. We define $f : V(Q_n) \to \{1, 2, 3, \cdots, 4q-1 = 16n-1\}$ as follows:

$$f(u_i) = \begin{cases} 6i-5, & 1 \leq i \leq n+1 \text{ and } i \text{ is odd} \\ 16n-10i+11, & 1 \leq i \leq n+1 \text{ and } i \text{ is even} \end{cases}$$

$$f(v_i) = \begin{cases} 16n-10i+9, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 6i-3, & 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

$$f(w_i) = \begin{cases} 6i-1, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 16n-10i+3, & 1 \leq i \leq n \text{ and } i \text{ is even.} \end{cases}$$

The induced edge labels are given by

$$f^*(u_iu_{i+1}) = \begin{cases} 8n-8i+3, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 8n-8i+5, & 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

$$f^*(v_iw_i) = \begin{cases} 8n-8i+5, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 8n-8i+3, & 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

$$f^*(u_iv_i) = 8n-8i+7, \quad 1 \leq i \leq n \text{ and}$$

$$f^*(u_iw_{i-1}) = 8n-8i+9, \quad 2 \leq i \leq n+1.$$

Thus, f is a skolem difference odd mean labeling and hence Q_n is a skolem difference odd mean graph. □

For example, a skolem difference odd mean labeling of Q_5 is shown in Fig.6.

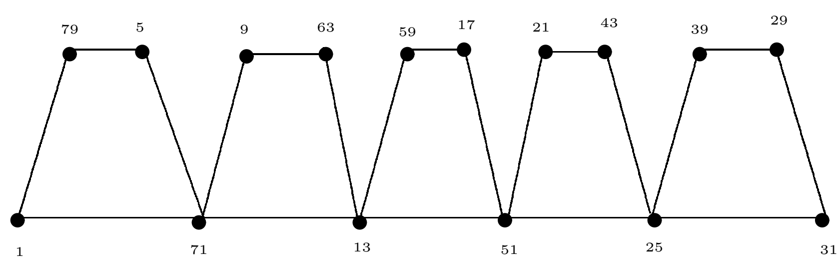


Fig.6

Theorem 2.7 *The ladder $L_n = P_n \times K_2$ is a skolem difference odd mean graph.*

Proof Let $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$ be the vertices of L_n and $E(L_n) = \{u_iv_i : 1 \leq i \leq n\} \cup \{u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{v_iv_{i+1} : 1 \leq i \leq n-1\}$. Define $f : V(L_n) \to \{1, 2, 3, \cdots, 4q-1 = 12n-9\}$ as follows:

$$f(u_i) = \begin{cases} 4i-3, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 12n-8i-1, & 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

$$f(v_i) = \begin{cases} 12n-8i-1, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 4i-3, & 1 \leq i \leq n \text{ and } i \text{ is even.} \end{cases}$$

For the vertex labeling f, the induced edge labeling f^* is given as follows:

$$f^*(u_iv_i) = 6n-6i+1, \quad 1 \leq i \leq n$$

$$f^*(u_iu_{i+1}) = \begin{cases} 6n-6i-3, & 1 \leq i \leq n-1 \text{ and } i \text{ is odd} \\ 6n-6i-1, & 1 \leq i \leq n-1 \text{ and } i \text{ is even} \end{cases}$$

$$f^*(v_iv_{i+1}) = \begin{cases} 6n-6i-1, & 1 \leq i \leq n-1 \text{ and } i \text{ is odd} \\ 6n-6i-3, & 1 \leq i \leq n-1 \text{ and } i \text{ is even.} \end{cases}$$

Then, f is a skolem difference odd mean labeling and hence L_n is a skolem difference odd mean graph. $\square$

For example, a skolem difference odd mean labeling of $L_8 = P_8 \times K_2$ is shown in Fig.7.

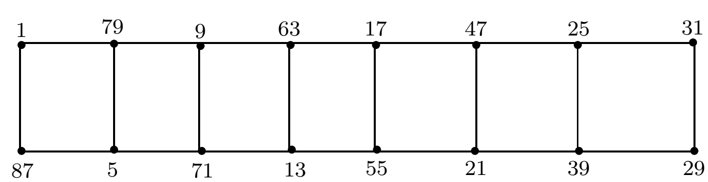


Fig.7

Theorem 2.8 *$L_n \odot K_1$ is a skolem difference odd mean graph.*

Proof Let L_n be the ladder. Let G be the graph obtained by joining a pendant edge to each vertex of the ladder. let $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$ be the vertices of the ladder. For $1 \leq i \leq n$, let u_i' and v_i' be the new vertices made adjacent with u_i and v_i respectively. The graph G has $4n$ vertices and $5n-2$ edges.

Define $f : V(G) \to \{1, 2, \cdots, 4q - 1 = 20n - 9\}$ by

$$f(u_i) = \begin{cases} 20n - 8i - 1, & 1 \le i \le n \text{ and } i \text{ is odd} \\ 12i - 7, & 1 \le i \le n \text{ and } i \text{ is even} \end{cases}$$

$$f(v_i) = \begin{cases} 12i - 7, & 1 \le i \le n \text{ and } i \text{ is odd} \\ 20n - 8i - 1, & 1 \le i \le n \text{ and } i \text{ is even} \end{cases}$$

$$f(u_i') = \begin{cases} 12i - 11, & 1 \le i \le n \text{ and } i \text{ is odd} \\ 20n - 8i - 5, & 1 \le i \le n \text{ and } i \text{ is even} \end{cases}$$

$$f(v_i') = \begin{cases} 20n - 8i - 5, & 1 \le i \le n \text{ and } i \text{ is odd} \\ 12i - 11 & 1 \le i \le n \text{ and } i \text{ is even} \end{cases}$$

For the vertex labeling f, the induced edge labeling f^* is obtained as follows:

$$f^*(u_iu_{i+1}) = \begin{cases} 10n - 10i - 3, & 1 \le i \le n - 1 \text{ and } i \text{ is odd} \\ 10n - 10i - 1, & 1 \le i \le n - 1 \text{ and } i \text{ is even} \end{cases}$$

$$f^*(v_iv_{i+1}) = \begin{cases} 10n - 10i - 1, & 1 \le i \le n - 1 \text{ and } i \text{ is odd} \\ 10n - 10i - 3, & 1 \le i \le n - 1 \text{ and } i \text{ is even} \end{cases}$$

$$f^*(u_iu_i') = \begin{cases} 10n - 10i + 5, & 1 \le i \le n \text{ and } i \text{ is odd} \\ 10n - 10i + 1, & 1 \le i \le n \text{ and } i \text{ is even} \end{cases}$$

$$f^*(v_iv_i') = \begin{cases} 10n - 10i + 1, & 1 \le i \le n \text{ and } i \text{ is odd} \\ 10n - 10i + 5, & 1 \le i \le n \text{ and } i \text{ is even.} \end{cases}$$

Thus, $L_n \odot K_1$ is a skolem difference odd mean labeling and hence $L_n \odot K_1$ is a skolem difference odd mean graph. □

For example, a skolem difference odd mean labeling of $L_7 \odot K_1$ is shown in Fig.8.

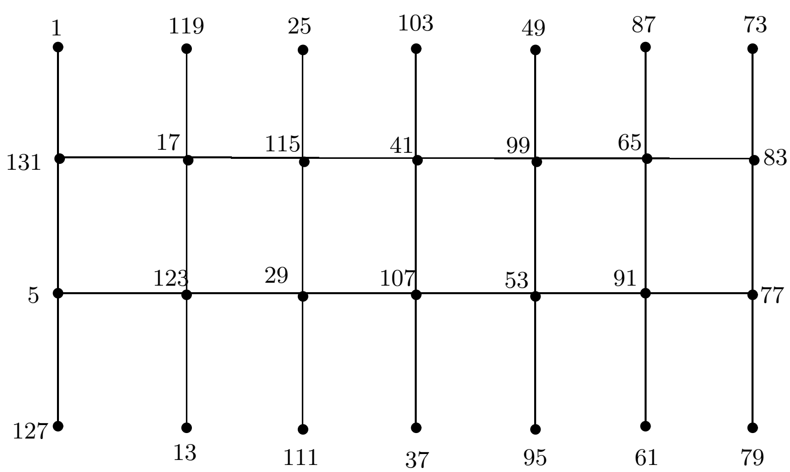


Fig.8

Theorem 2.9 *Let the path $G_1 = (p_1, q_1)$ and the star $G_2 = (p_2, q_2)$ have skolem difference odd mean*

labeling f and g respectively. Let u be the end vertex of G_1 and v be the central vertex of G_2 such that $f(u) = 1$ and $g(v) = 1$. Then the graph $(G_1)_f * (G_2)_g$ obtained from G_1 and G_2 by identifying the vertices u and v is also skolem difference odd mean.

Proof Let $V(G_1) = \{u, u_i : 1 \leq i \leq p_1 - 1\}$ and $V(G_2) = \{v, v_i : 1 \leq i \leq p_2 - 1\}$. Then the graph $(G_1)_f * (G_2)_g$ has $p_1 + p_2 - 1$ vertices and $q_1 + q_2$ edges.

Define $h : V((G_1)_f * (G_2)_g) \rightarrow \{1, 2, 3, \cdots, 4(q_1 + q_2) - 1\}$ as follows:

$$h(u_i) = f(u_i), 1 \leq i \leq p_1 - 1$$
$$h(u) = f(u) = g(v) \text{ and}$$
$$h(v_i) = g(v_i) + 2(p_1 + q_1 - 1), 1 \leq i \leq p_2 - 1.$$

Then the induced edge labels of G_1 are $1, 3, 5, \ldots, 2q_1 - 1$ and that of G_2 are

$$2q_1 + 1, 2q_1 + 3, \cdots, 2(q_1 + q_2) - 1.$$

Hence, the graph $(G_1)_f * (G_2)_g$ obtained from G_1 and G_2 by identifying the vertices u and v is a skolem difference odd mean graph. □

For example, a skolem difference odd mean labeling of G_1, G_2 and $G_1 * G_2$ are shown in Fig.9.

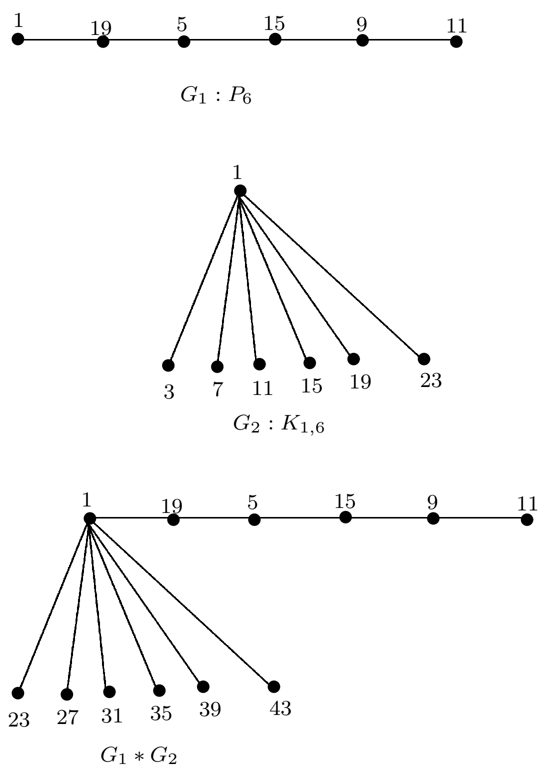


Fig.9

Theorem 2.10 *The graph $K_{1,n} \odot K_1$ is skolem difference odd mean for all $n \geq 1$.*

Proof Let G be the graph $K_{1,n} \odot K_1$ obtained from the star $K_{1,n}$ with vertices $u_0, u_1, u_2, \cdots, u_n$ by joining a vertex v_i to $u_i, 0 \leq i \leq n$.

Let $V(G) = \{u_0, u_i, v_0, v_i : 1 \leq i \leq n\}$ and $E(G) = \{u_0v_0, u_0u_i, u_iv_i : 1 \leq i \leq n\}$. The graph G has $2n+2$ vertices and $2n+1$ edges. Define $f : V(G) \to \{1, 2, \cdots, 4q-1 = 8n+3\}$ as follows:

$$f(u_0) = 1$$
$$f(u_i) = 4n + 4i - 1, 1 \leq i \leq n$$
$$f(v_0) = 8n + 3$$
$$f(v_i) = 8i - 3, 1 \leq i \leq n.$$

For the vertex labeling f, the induced edge labeling f^* in given as follows:

$$f^*(u_0v_0) = 4n + 1$$
$$f^*(u_0u_i) = 2n + 2i - 1, 1 \leq i \leq n$$
$$f^*(u_iv_i) = 2n - 2i + 1, 1 \leq i \leq n.$$

Then, f is a skolem difference odd mean labeling and hence G is a skolem difference odd mean graph. $\square$

For example, a skolem difference odd mean labeling of $K_{1,7} \odot K_1$ is shown in Fig.10.

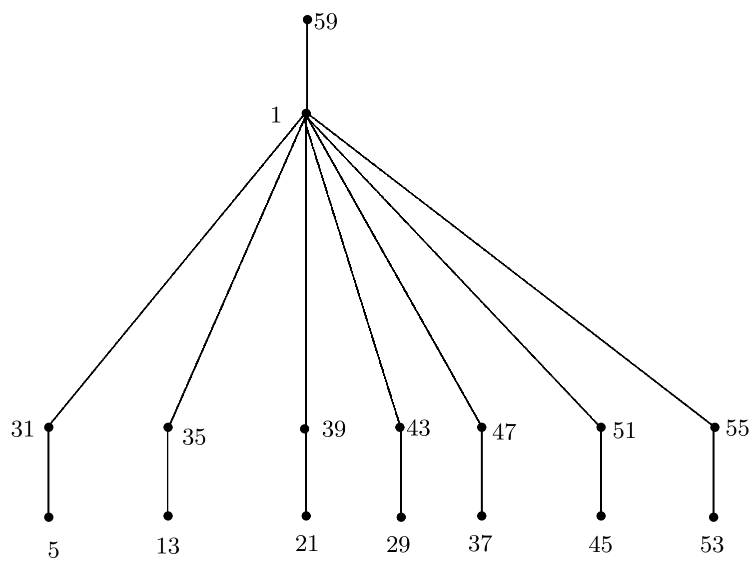


Fig.10

References

[1] F.Harary, *Graph Theory*, Addison-Wesley, Reading Mass., 1972.

[2] K.Manikam and M.Marudai, Odd mean labeling of graphs, *Bulletin of Pure and Applied Sciences*, **25E**(1) (2006), 149–153.

[3] K.Murugan and A.Subramanian, Skolem difference mean labeling of H-graphs, *International Journal of Mathematics and Soft Computing*, **1**(1) (2011), 115–129.

[4] A.Nagarajan and R.Vasuki, On the meanness of arbitrary path super subdivision of paths, *Australian J. Combin.*, **51** (2011), 41–48.

[5] S.Somasundaram and R.Ponraj, Mean labelings of graphs, *National Academy Science Letters*, **26** (2003), 210–213.

[6] R.Vasuki and A.Nagarajan, Meanness of the graphs $P_{a,b}$ and P_a^b, *International Journal of Applied Mathematics*, **22**(4) (2009), 663–675.

[7] R.Vasuki and S.Arokiaraj, On mean graphs, *International Journal of Mathematical Combinatorics*, **3** (2013), 22–34.

[8] R.Vasuki and A.Nagarajan, Odd mean labeling of the graphs $P_{a,b}, P_a^b$ and $P_{\langle 2a\rangle}^b$, *Kragujevac Journal of Mathematics*, **36**(1) (2012), 125–134.

[9] R.Vasuki and S.Arokiaraj, On odd mean graphs, *Journal of Discrete Mathematical Sciences and Cryptography*, (To appear).

[10] S.K.Vaidya and Lekha Bijukumar, Some new families of mean graphs, *Journal of Mathematics Research*, **2**(3) (2010), 169–176.

Math.Combin.Book Ser. Vol.3(2015), 99-104

Radio Number for Special Family of Graphs with Diameter 2, 3 and 4

M.Murugan

(School of Science, Tamil Nadu Open University, 577, Anna Salai, Chennai - 600 015, India)

E-mail: muruganganesan@yahoo.in

Abstract: A radio labeling of a graph G is an injective function [4] $f : V(G) \to \mathbb{N} \cup \{0\}$ such that for every $u, v \in V(G)$,

$$|f(u) - f(v)| \geqslant \operatorname{diam}(G) - d(u, v) + 1.$$

The span of f is the difference of the largest and the smallest channels used, that is,

$$\max_{u,v \in V(G)} \{f(u) - f(v)\}.$$

The radio number of G is defined as the minimum span of a radio labeling of G and denoted as $rn(G)$. In this paper, we present algorithms to get the radio labeling of special family of graphs like double cones, books and nC_4 with a common vertex whose diameters are 2,3 and 4 respectively.

Key Words: Radio labeling, radio number, channel assignment, distance two labeling.

AMS(2010): 05C78, 05C12, 05C15.

§1. Introduction

The channel assignment problem is a telecommunication problem in which our aim is to assign a channel (non-negative integer) to each TV or Radio station so that we do not have any interference in the communication. The level of interference between the TV or Radio stations correlates with the geographic locations of these stations. Earlier designers of TV networks considered close locations and very close locations so that the transmitters at the close locations receive different channels and the transmitters at very close locations are at least two apart for clear communication.

This can be modeled by graph models in which vertices represent the stations and two vertices are joined by an edge if they are very close and they are joined by a path of length 2 if they are close.

The mathematical abstraction of the above concept is distance two labeling or $L(2,1)$-labeling. A $L(2,1)$-labeling of a graph G is an assignment f from the vertex set $V(G)$ to the set of non-negative integers such that $|f(x) - f(y)| \geqslant 2$ if x and y are adjacent and $|f(x) - f(y)| \geqslant 1$ if x and y are at distance 2, for all x and y in $V(G)$.

Practically speaking the interference among channels may go beyond two levels. So we have to

[1]Received October 12, 2014, Accepted August 25, 2015.

extend the interference level from two to the largest possible - the diameter of the corresponding graph. So the concept of $L(2,1)$ labeling was generalized to radio labeling.

Radio labeling was originally introduced by G.Chartrand et.al., [1] in 2001. A radio labeling of a graph G is an injective function $f: V(G) \to \mathbb{N} \cup \{0\}$ such that for every $u, v \in V(G)$,

$$|f(u) - f(v)| \geqslant \mathrm{diam}(G) - d(u,v) + 1.$$

The span of f is the difference of the largest and the smallest channels used, that is,

$$\max_{u,v \in V(G)} \{f(u) - f(v)\}.$$

The radio number of G is defined as the minimum span of a radio labeling of G and denoted as $rn(G)$.

§2. Some Existing Results

(1) D.D.F. Liu and X. Zhu [2] have discussed the radio number for paths and cycles.

(2) D.D.F. Liu [3] has obtained lower bounds for the radio number of trees and characterized trees achieving this bound.

(3) Mustapha et al. [4] have discussed radio k-labeling for cartesian products of graphs.

(4) The radio labeling of cube and fourth power of cycles have been discussed by B. Sooryanarayana and P. Raghunath [5, 6].

(5) The radio labeling of k^{th} power of a path is discussed by P. Devadas Rao et al. [7]

(6) The radio number for the split graph and the middle graph of cycle C_n is discussed by S.K. Vaidya et al. [8].

In the survey of literature available on radio labeling, we found that only two types of problems are considered in this area till this date.

(1) To investigate bounds for the radio number of a graph;

(2) To completely determine the radio number of a graph.

§3. Results

In this section, we develop and justify algorithms to get Radio labeling of the family of graphs - double cones whose diameter is 2, books whose diameter is 3 and nC_4 with a common vertex whose diameter is 4.

We note that the concept of radio labeling and $L(2,1)$ labeling coincides when the diameter of the graph is 2. In general, to prove a labeling is a radio labeling, we have to prove that every $u, v \in V(G)$, $|f(u) - f(v)| \geqslant \mathrm{diam}(G) - d(u,v) + 1$.

Double Cones are graphs obtained by joining two isolated vertices to every vertex of C_n. A Double Cone on $n+2$ vertices is denoted by $CO_{n+2} = C_n + 2K_1$. We note that CO_{n+2} has two vertices with maximum degree n.

Algorithm 1:

Input : A double cone $CO_{n+2} = C_n + 2K_1$, $n \geqslant 5$.

Output : Radio labeling of CO_{n+2}.

Step 1: Arrange the vertices of C_n as $v_1, v_2, \ldots, v_n$ and let u_1, u_2 denotes $2K_1$.

Step 2: First label v_is of odd i and then label v_is of even i starting from the label 3 and ending with $n+2$ consecutively.

Step 3 : Label u_1 with 0 and u_2 with 1.

The justification of the desired output is proved in Theorem 3.1.

Theorem 3.1 *For the double cone CO_{n+2}, $n \geqslant 5$, $rn(CO_{n+2}) = n+2$.*

Proof Consider the double cone CO_{n+2}, $n \geqslant 5$. Let $V(Cn) = \{v_1, v_2, \ldots, v_n\}$ and u_1, u_2 denotes $2K_1$. Define $f : V(CO_{n+2}) \to \mathbb{N} \cup \{0\}$ such that

$$\begin{aligned}
f(u_1) &= 0 \\
f(u_2) &= 1 \\
f(v_1) &= 3 \\
f(v_i) &= f(v_1) + \frac{i-1}{2} \text{ if } i \text{ is odd and } i > 1. \\
f(v_2) &= f(v_1) + \left\lceil \frac{n}{2} \right\rceil \\
f(v_i) &= f(v_1) + \left\lceil \frac{n}{2} \right\rceil + \frac{i-2}{2} \text{ if } i \text{ is even and } i > 2.
\end{aligned}$$

When $d(v_i, v_j) = 1$, we have $|f(v_i) - f(v_j)| = \lceil \frac{n}{2} \rceil$ or $\lceil \frac{n}{2} \rceil - 1$, $n-1$. That is, when $d(v_i, v_j) = 1$, we have $|f(v_i) - f(v_j)| \geqslant 2$. When $d(v_i, v_j) = 2$, we have $|f(v_i) - f(v_j)| \geqslant 1$ since $f(v_i)$s are distinct. Since $f(v_i) \geqslant 3$ and $f(u_1) = 0$, we have $|f(v_i) - f(u_1)| \geqslant 3$. Since $f(v_i) \geqslant 3$ and $f(u_2) = 1$, we have $|f(v_i) - f(u_2)| \geqslant 2$. Hence f is a radio labeling and $rn(CO_{n+1}) \leqslant n+2$.

Since CO_{n+2} has two vertices u_1 and u_2 with maximum degree n and that too, they are adjacent with the same vertices $v_1, v_2, \ldots, v_n$, the consecutive $n+2$ labels $0, 1, 2, \ldots, n+1$ are not sufficient to produce a radio-labeling. Therefore, $rn(CO_{n+2}) \geqslant n+2$ and hence $rn(CO_{n+2}) = n+2$. □

A book B_n is the product of the star $K_{1,n}$ with K_2. A book B_n has $p = 2n+2$ vertices, $q = 3n+1$ edges, n pages, maximum degree $n+1$ and diameter 3. Next we present an algorithm to get a radio labeling of a book B_n.

Algorithm 2:

Input : A book B_n, $n \geqslant 4$, with p vertices.

Output : Radio labeling of B_n.

Step 1: Let the vertices of the n^{th} page of B_n be v, v_n, w_n and w where v and w lie on the common edge vw.

Step 2: Label v with the label $p+1$ and w with the label 0.

Step 3: Label v_i with $2i$, $i = 1, 2, \cdots, n$.

Step 4: Label w_i with $f(v_i) + 5$, $i = 1, 2, \ldots, n-2$ and w_{n-1} with the label 3 and w_n with the label 5, where $f(v_i)$ is the label of v_i.

The justification of the desired output is proved in Theorem 3.2.

Theorem 3.2 *For a Book B_n, $n \geqslant 4$, $rn(B_n) = p+1$.*

Proof Consider a book B_n. A book B_n has $p = 2n+2$ vertices, $q = 3n+1$ edges, n pages, maximum degree $n+1$ and diameter 3. Let the vertices of the n^{th} page of B_n be v, v_n, w_n and w where v and w lie on the common edge vw.

Define $f : V(B_n) \to \mathbb{N} \cup \{0\}$ such that

$$\begin{aligned} f(v) &= p+1 \\ f(w) &= 0 \\ f(v_i) &= 2i, \quad i = 1,2,\cdots,n \\ f(w_i) &= f(v_i)+5, \quad i = 1,2,\cdots,n-2 \\ f(w_{n-1}) &= 3 \\ f(w_n) &= 5 \end{aligned}$$

We note that $d(v, v_i) = 1$, $d(w, w_i) = 1$, $d(v_i, w_i) = 1$, $i = 1, 2, \ldots, n$ and $d(v, w) = 1$. Now,

$$\begin{aligned} |f(v) - f(v_i)| &= p+1-2i = 2n+2+1-2i = 2n+3-2i \geqslant 3, \quad i = 1,2,\cdots,n. \\ |f(w) - f(w_i)| &= f(v_i)+5 = 2i+5 \geqslant 3, \quad i = 1,2,\cdots,n-2. \\ |f(w) - f(w_{n-1})| &= 3. \\ |f(w) - f(w_n)| &= 5. \end{aligned}$$

Consider $f(v_i) - f(w_i) = 2i+5-2i = 5, \quad i = 1,2,\cdots,n-2.$

$$\begin{aligned} |f(v_{n-1}) - f(w_{n-1})| &= 2n-2-3 = 2n-5 \geqslant 3. \\ |f(v_n) - f(w_n)| &= 2n-5 \geqslant 3. \\ \text{Also} \quad |f(v) - f(w)| &= p+1 = 2n+3. \end{aligned}$$

Thus, $|f(s_i) - f(t_i)| \geqslant 3$ when $d(s_i, t_i) = 1$, $s_i, t_i \in V(B_n)$. We note that $d(v, w_i) = 2$, $d(w, v_i) = 2$, $i = 1,2,\cdots,n$, $d(v_i, v_j) = 2$, $i \neq j$, $i,j = 1,2,\cdots,n$ and $d(w_i, w_j) = 2$, $i \neq j$, $i,j = 1,2,\cdots,n$.

When $d(v, w_i) = 2$, $i = 1,2,\cdots,n$,

$$\begin{aligned} |f(v) - f(w_i)| &= p+1-2i-5 = 2n+2+1-2i-5 = 2n-2-2i \geqslant 2, \\ & \qquad\qquad\qquad i = 1,2,\cdots,n-2. \\ |f(v) - f(w_{n-1})| &= p+1-3 = p-2 \geqslant 2 \\ |f(v) - f(w_n)| &= p+1-5 = p-4 \geqslant 2 \end{aligned}$$

When $d(w, v_i) = 2$, $i = 1,2,\cdots,n$, $|f(w) - f(v_i)| = 2i \geqslant 2$, $i = 1,2,\cdots,n$. When $d(v_i, v_j) = 2$, $i \neq j$, $i,j = 1,2,\cdots,n$, consider for $i > j$, $|f(v_i) - f(v_j)| = 2i - 2j \geqslant 2$. When $d(w_i, w_j) = 2$, $i \neq j$, $i,j = 1,2,\cdots,n-2$, consider for $i > j$, $|f(w_i) - f(w_j)| = |f(v_i) - f(v_j)| = 2i - 2j \geqslant 2$. Since $f(w_{n-1}) = 3$ and $f(w_n) = 5$, $|f(w_i) - f(w_j)| \geqslant 2$, for all $i,j = 1,2,\cdots,n$. Thus, $|f(s_i) - f(t_i)| \geqslant 2$ when $d(s_i, t_i) = 2$, $s_i, t_i \in V(B_n)$.

Since all the vertex labels are distinct, we have,

$$|f(s_i) - f(t_i)| \geqslant 1 \text{ when } d(s_i, t_i) = 3, \; s_i, t_i \in V(B_n).$$

Hence f is a radio labeling and $rn(B_n) \leqslant p+1$.

Since there are p vertices, there should be p distinct labels. The p labels $0, 1, 2, \cdots, p-1$ are not

sufficient to produce a radio labeling of the Book B_n.

Suppose they produce a radio labeling, we note that B_n has 2 adjacent vertices v and w such that each is adjacent with a different set of $\frac{p-2}{2}$ vertices, say, A and B respectively such that

(i) each vertex in A should have label difference at least 3 with v and among themselves they should have label difference at least 2;

(ii) each vertex in B should have label difference at least 3 with w and among themselves they should have label difference at least 2 and

(iii) the label difference of v and w should be 3.

This is not possible with p consecutive non-negative integers $0, 1, 2, \cdots, p-1$. Therefore B_n should have more than these p labels $0, 1, 2, \cdots, p-1$. Since we have 2 partitions of the vertex set of B_n, say, $A \cup \{v\}$ and $B \cup \{w\}$ with the above three properties, B_n should have at least 2 more labels to have a radio labeling. Therefore, $rn(B_n) \geqslant p+1$. Hence $rn(B_n) = p+1$. □

Now we consider the graph, the collection of n copies of C_4s, all of which have a common vertex, that is, nC_4 with a common vertex.

Algorithm 3:

Input : nC_4 with a common vertex, $n \geqslant 3$.

Output : Radio labeling of the graph, nC_4 with a common vertex.

Step 1: Let the common vertex be u and the vertices of the i^{th} cycle be u, v_{i_1}, v_{i_2} and v_{i_3}, $i = 1, 2, \cdots, n$.

Step 2: Label the common vertex u with $7n+2$.

Step 3: Label v_{i_2} with label $i-1$, $i = 1, 2, \cdots, n$.

Step 4: Arrange the remaining vertices as $v_{11}, v_{13}, v_{21}, v_{23}, \cdots, v_{n_1}, v_{n_3}$. Label these vertices starting from $n+1, n+4, n+7, \ldots, 7n-2$.

The justification of the desired output is proved in Theorem 3.3.

Theorem 3.3 *Let G denotes nC_4 with a common vertex. Then $rn(G) = 7n+2$, $n \geqslant 3$.*

Proof Consider the graph, nC_4 with a common vertex. Let the common vertex be u and the vertices of the i^{th} cycle be u, v_{i_1}, v_{i_2} and v_{i_3}, $i = 1, 2, \cdots, n$.

Define $f : V(G) \to \mathbb{N} \cup \{0\}$ such that

$$\begin{aligned} f(u) &= 7n+2 \\ f(v_{i_2}) &= i-1, \quad i = 1, 2, \cdots, n \\ f(v_{i_1}) &= n+1+(i-1)6, \quad i = 1, 2, \cdots, n \\ f(v_{i_3}) &= n+4+(i-1)6, \quad i = 1, 2, \cdots, n \end{aligned}$$

We note that $d(u, v_{ij}) = 1$, $i = 1, 2, \cdots, n$ and $j = 1, 3$. Also $d(v_{i_1}, v_{i_2}) = 1$ and $d(v_{i_3}, v_{i_2}) = 1$, $i = 1, 2, \cdots, n$. Now, $|f(u) - f(v_{i_1}) = 7n+2-n-1-(i-1)6 = 6n+1-(i-1)6 = 6n+7-6i \geqslant 4, i = 1, 2, \cdots, n$ and $|f(u) - f(v_{i_3}) = 7n+2-n-4-(i-1)6 = 6n-2-(i-1)6 = 6n+4-6i \geqslant 4, i = 1, 2, \cdots, n$.

Consider

$$|f(v_{i_1}) - f(v_{i_2})| = n+1+(i-1)6-(i-1) = n+1+(i-1)5 \geqslant 4, \ i = 1, 2, \cdots, n.$$

Consider

$$|f(v_{i_3})-f(v_{i_2})| = n+4+(i-1)6-(i-1) = n+4+(i-1)5 \geqslant 4,\ \ i=1,2,\cdots,n.$$

We note that, for $i=1,2,\ldots,n$ and $j=2$, $d(u,v_{ij})=2$ and for i, $\alpha=1,2,\cdots,n$ and $j,k=1,3$, $d(v_{ij},v_{\alpha k})=2$ $(i,\alpha,j,k$ all are not equal).

Since $f(u)=7n+2$ and for $i=1,2,\cdots,n$ and $j=2$, $f(v_{ij})=i-1$, we have,

$$|f(u)-f(v_{ij})| = 7n+2-i+1 = 7n+3-i \geqslant 3,\ \ \ i=1,2,\cdots,n \text{ and } j=2.$$

Now, since, for $i=1,2,\cdots,n$, $f(v_{i_1})=n+1+(i-1)6$ and $f(v_{i_3})=n+4+(i-1)6$, we have, for $i,\alpha=1,2,\cdots,n$ and $j,k=1,3$, $(i,\alpha,j,k$ all are not equal), $|f(v_{ij})-f(v_{\alpha k})\geqslant 3$.
Next, we note that, for $i=1,2,\cdots,n$, $d(v_{i_2},v_{j_k})=3$ where $j=1,2,\cdots,n$ & $j\neq i$ and $k=1,3$. In this case, $|f(v_{j_1})-f(v_{i_2})=n+1+(j-1)6-(i-1)=n+6j-i-4\geqslant 2$. Also, $|f(v_{j_3})-f(v_{i_2})|=n+4+(j-1)6-(i-1)=n+6j-i-1\geqslant 2$. Finally, for $i\neq j$, $d(v_{i_2},v_{j_2})=4$. Since all the labels of v_{i_2}, $i=1,2,\cdots,n$ are distinct, $|f(v_{i_2})-f(v_{j_2})|\geqslant 1$. Hence f is a radio labeling and $rn(G)\leqslant 7n+2$.

Now we find the optimal value of $rn(G)$. The minimum possible label of u is 0. Since the distance between u and v_{11} is 1, the minimum label we can use at v_{11} is at least 4. Since the vertices $v_{13},v_{21},v_{23},v_{31},v_{33},\cdots,v_{n_1},v_{n_3}$ are at a distance 2 from v_{11}, each of this should have a label difference 3 with v_{11} and among themselves. And so the minimum range of the labels for these vertices is $\{7,10,\cdots,6n+1\}$. So, without loss of generality,we assume that the minimum label v_{n_3} should receive is $6n+1$. There are n more vertices, namely, $v_{12},v_{22},\cdots,v_{n_2}$. We cannot use the label $6n+2$ to any of these vertices because these vertices are at a distance 3 from v_{n_3} and hence the label difference at these vertices should be at least 2. So the minimum label we can use to any one of these vertices is $6n+3$ and there are n such vertices. Therefore the minimum label we required to label all these vertices is $6n+1+(n+1)=7n+2$.

Therefore, the minimum label which can give a radio labeling for the graph G is $7n+2$. That is, $rn(G)\geqslant 7n+2$. Hence, $rn(G)=7n+2$. □

References

[1] Gary Chartrand, David Erwin, Frank Harary and Phing Zhang, Radio labelings of graphs, *Bulletin of Inst. Combin. Appl.*, 33, (2001), 77-85.

[2] Daphne Der-Fen Liu and Xuding Zhu, Multilevel distance labelings for paths cycles, *SIAM J. Discrete Math.*, 19, (2005), 610-621.

[3] Daphne Der-Fen Liu, Radio number for trees, *Discrete Mathematics*, 308, (7), (2007), 1153-1164.

[4] Mustapha Kchilkech, Riadh Khennoufs and Olivier Togni, Radio k-Labelings for Cartesian Products of Graphs, *Discussiones Mathematicae Graph Theory*, 28(1) (2008), 165-178.

[5] B.Sooryanarayana and P.Raghunath, On the radio labeling of cube of a cycle, *Far East J. Appl. Math.*, 29 (1), (2007), 113-147.

[6] B.Sooryanarayana and P.Raghunath, On the radio labeling of fourth power of a cycle, *Journal of Applied Mathematical Analysis and Applications*, 3(2), (2007), 195-228.

[7] P.Devadas Rao, B.Sooryanarayana and Chandru Hegde, On the radio labeling of k^{th} power of a path, *Journal of Graphs and Combinatorics* , Submitted.

[8] S.K.Vaidya and P.L.Vihol, Radio labeling for some cycle related graphs, *International Journal of Mathematics and Soft Computing*, Vol.2. No.2 (2012), 11-24.

Math.Combin.Book Ser. Vol.3(2015), 105-115

Vertex-to-Edge-set Distance Neighborhood Pattern Matrices

Kishori P.Narayankar and Lokesh S. B.

(Department of Mathematics, Mangalore University, Mangalagangothri-574 199, India)

Veena Mathad

(Department of Studies in Mathematics, University of Mysore, Manasagangothri-570 006, India)

E-mail: kishori_pn@yahoo.co.in, sbloki83@gmail.com, veena_mathad@rediffmail.com

Abstract: The vertex to edge set (VTES) distance $d_1(u,e)$ from a vertex $u \in V(G)$ to an edge $e \in E(G)$ is the number of edges on $(u-e)$ path. For each $u \in V(G)$ define $N_j^M[u] = \{e \in M \subseteq V(G) : d(u,e) = j, \text{ where } 0 \leq j \leq d_1(G)\ \}$ and a non-negative integer matrix $D_1^M(G) = (|N_j^M[u]|)$ of order $V(G) \times ((d_1(G)+1)$ called the VTES-M-distance neighborhood pattern (M-dnp) matrix of G. If $f_M : u \longmapsto f_M(u)$ is an injective function, where $f_M(u) = \{j : N_j^M[u] \neq \emptyset\}$, then the set M is a VTES-distance pattern distinguishing (M-dpd) set of G and G is a VTES-dpd-graph. This paper is a study of VTES M-dnp-matrices of a VTES-dpd-graph.

Key Words: Distance (in Graph), vertex-to-edge-set distance-pattern distinguishing sets, VTES-distance neighborhood pattern matrix.

AMS(2010): 05C12, 05C50.

§1. Introduction

For all terminology which are not defined in this paper, we refer the reader to F.Harary [6]. Unless mentioned otherwise, all the graphs considered in this paper are nontrivial, simple, finite and connected.

Distance between two elements (vertex to vertex, vertex to edge, edge to vertex, and edge to edge) in graphs is already defined in the literature (refer [9]), but here we are using vertex to edge-distance. For subsets $S, T \subseteq V(G)$, and any vertex v, let $d(v,S) = min\{d(v,u) : u \in S\}$ and $d(S,T) = min\{d(x,y) : x \in S, y \in T\}$. In particular, if $f = xy$ is an edge in G, then the vertex to edge distance between v and f is given by $d(v,f) = min\{d(v,x), d(v,y)\}$ [9].

A study of these sets is expected to be useful in a number of areas of application such as facility location [5] and design of indices of "quantitative structure activity relationships" (QSAR) in chemistry ([2], [8]).

Definition 1.1([9]) *For any vertex v in a connected graph G, the vertex-to-edge eccentricity $\epsilon(v)$ of v is $\epsilon(v) = max\,\{d(v,e) : e \in E(G)\}$. The vertex-to-edge diameter $d_1(G) = max\,\{\epsilon(v)\}$ and the vertex-to-edge radius $r_1(G) = min\,\{\epsilon(v)\}$. A vertex v for which $\epsilon(v)$ is minimum is called a vertex-to-edge central vertex of G and the set of all vertex-to-edge central vertices of G is the vertex-to-edge center $C_1(G)$ of G. Any edge e for which $\epsilon(v) = d(v,e)$ called an eccentric edge of v.*

[1]Received January 29, 2015, Accepted August 28, 2015.

The vertex-to-vertex eccentricities and the vertex-to-edge eccentricities of the vertices of graphs G and H in Fig.1 are given in the Table 1.1 and Table 1.2, respectively.

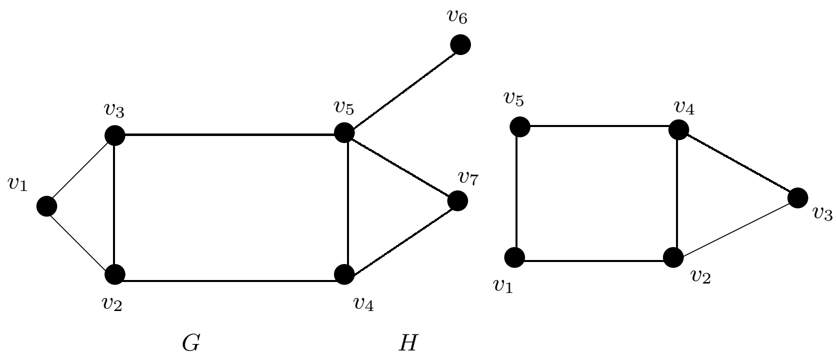


Fig 1

v	v_1	v_2	v_3	v_4	v_5	v_6	v_7
$e(v)$	3	3	2	2	2	3	3
$\epsilon(v)$	2	2	2	2	2	3	2

Table 1.1

v	v_1	v_2	v_3	v_4	v_5
$e(v)$	2	2	2	2	2
$\epsilon(v)$	2	1	2	1	2

Table 1.2

Definition 1.2 *Let $G = (V, E)$ be a given connected simple (p,q)-graph, $M \subseteq E(G)$ and for each $u \in V(G)$, let $f_M(u) = \{d(u, e) : e \in M\}$ be the distance-pattern of u with respect to M. If f_M is injective then the set M is a distance-pattern distinguishing set (or, a "VTES-dpd-set" in short) of G and G is a VTES-dpd-graph. If $f_M(u) - \{0\}$ is independent of the choice of u in G then M is an open distance-pattern uniform (or, VTES-odpu) set of G and G is called an VTES-odpu-graph. The minimum cardinality of a VTES-dpd-set (VTES-odpu-set) in G, if it exists, is the VTES-dpd-number(VTES-odpu-number) of G and it is denoted by $\rho(G)$.*

For an arbitrarily fixed vertex u in G and for any nonnegative integer j, we let $N_j[u] = \{e \in E(G) : d(u, e) = j\}$. Clearly, $|N_0[u]| = \{deg(u)\}$, $\forall u \in V(G)$ and $N_j[u] = V(G) - V(\xi_u)$ whenever j exceeds the vertex-to-edge eccentricity $\epsilon(u)$ of u in the component ξ_u to which u belongs. Thus, if G is connected then, $N_j[u] = \phi$ if and only if $j > \epsilon(u)$. If G is a connected graph then the vectors $\bar{u} = (|N_0[u]|, |N_1[u]|, |N_2[u]|, \cdots, |N_{\epsilon(u)}[u]|)$ associated with $u \in V(G)$ can be arranged as a $p \times (d_{1G} + 1)$ matrix D_{1G} whose entries are nonnegative integers given by

$$\begin{pmatrix} |N_0[v_1]| & |N_1[v_1]| & |N_2[v_1]| & \ldots & |N_{\epsilon(v_1)}[v_1]| & 0 & 0 & 0 \\ |N_0[v_2]| & |N_1[v_2]| & |N_2[v_2]| & \ldots & \ldots & |N_{\epsilon(v_2)}[v_2]| & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ |N_0[v_p]| & |N_1[v_p]| & |N_2[v_p]| & \ldots & \ldots & \ldots & \ldots & |N_{\epsilon(v_p)}[v_p]| \end{pmatrix}$$

where d_{1G} denotes the vertex-toedge diameter of G; we call D_{1G} *VTES-distance neighborhood pattern matrix (or, VTES-dnp-matrix)* of G. For a VTES-dnp-matrix the following observations are immediate.

Observation 1.3 Entries in the first column of D_{1G} are nonzero entries.

Observation 1.34 In each row of D_{1G}, entry zero will be after some nonzero entries. Zero entries may or may not be present in rows.

Observation 1.5 The entries in the first column of D_{1G} correspond to the degrees of the corresponding vertices in G.

Proposition 1.6 *For each $u \in V(G)$ of a connected graph G, $\{N_j[u] : N_j[u] \neq \phi,\ 0 \leq j \leq d_{1G}\}$ gives a partition of $E(G)$.*

Proof If possible, let $e \in N_j[u] \cap N_k[u]$, for some $e \in E(G)$ and $u \in V(G)$. Then $d(u,e) = j$ and $d(u,e) = k$, and hence $j = k$. Therefore,$N_j[u] \cap N_k[u] = \phi$ for any (j,k) with $j \neq k$. Now, clearly, $\bigcup_{j=o}^{d_{1G}} N_j[u] \subseteq E(G)$. Also, for any $e \in E(G)$, since G is connected, $d(u,e) = k$, for some $k \in \{0,1,2,\cdots,d_{1G}\}$. That is, $e \in N_k[u]$ for some $k \in \{0,1,2,\cdots,d_{1G}\}$ which implies $E(G) \subseteq \bigcup_{j=o}^{d_{1G}} N_j[u]$. Hence $\bigcup_{j=o}^{d_{1G}} N_j[u] = E(G)$. $\square$

Corollary 1.7 *Each row of the VTES-dnp-matrix D_{1G} of a graph G is the partition of $|E(G)|$. Hence, sum of the entries in each row of the VTES-dnp-matrix D_{1G} of a graph G is equal to the number of edges of G.*

§2. M-VTES-Distance Neighborhood Pattern Matrix of a Graph

Given an arbitrary nonempty subset $M \subseteq E(G)$ of G and for each $u \in V(G)$, define $N_j^M[u] = \{e \in M : d(u,e) = j\}$; clearly then $N_j^{E(G)}[u] = N_j[u]$. One can define the M-VTES-eccentricity of u as the largest integer for which $N_j^M[u] \neq \phi$ and the $p \times (d_{1G}+1)$ nonnegative integer matrix $D_{1G}^M = (|N_j^M[u]|)$ is called the M-VTES-distance neighborhood pattern (or, M-VTES-dnp) matrix of G. D_G^{*M} is obtained from D_{1G}^M by replacing each nonzero entry by 1.

B.D.Acarya ([1]) defined VTES-dnp matrix of any graph and in particular, M-VTES-dnp matrix of VTES-dpd-graph as follows:

Definition 2.1 *Let $G = (V,E)$ be a given connected simple (p,q)-graph, $M(\neq \phi) \subseteq E(G)$ and $u \in V(G)$. Then, the M-VTES-distance-pattern of u is the set $f_M(u) = \{d(u,e) : e \in M\}$. Clearly, $f_M(u) = \{j : N_j^M[u] \neq \phi\}$. Hence, in particular, if $f_M : u \longmapsto f_M(u)$ is an injective function, then the set M is a VTES-distance-pattern distinguishing set (or, a "VTES-dpd-set" is short) of G and if $f_M(u) - \{0\}$ is independent of the choice of u in G then M is a VTES-open distance-pattern uniform (or, VTES-odpu) set of G. A graph G with a VTES-dpd-set(VTES-odpu-set) is called a VTES-dpd-(VTES-odpu)-graph.*

Following are some interesting results on M-VTES-dnp matrix of a connected nontrivial graph G.

Observation 2.2 Both D_{1G}^M and D_{1G}^{*M} do not admit null rows.

Proposition 2.3 *For each $u \in V(G)$,*

$$N_0^M[u] = \begin{cases} M \cap D_u, & if \quad M \cap D_u \neq \phi; \\ \emptyset, & if \quad M \cap D_u = \phi, \end{cases}$$

where $D_u = \{e_i : 1 \leq i \leq degu$ and u is adjacent to $e_i\}$.

Therefore, the entries in the first column of D_{1G}^M are zero or an integer $k, 1 \leq k \leq degu$ and the entries in the first column of D_{1G}^{*M} are either 0 or 1.

Remark 2.4 It should be noted that Observation is not true in the case of D_{1G}^{*M}.

Remark 2.5 For a graph $G \cong C_n$,

vertex-toedge diameter $d_{1G} = \begin{cases} \frac{n-2}{2} & if \quad \text{n} \quad is \quad even \quad integer \\ \frac{n-1}{2} & if \quad \text{n} \quad is \quad odd \quad integer \end{cases}$

Remark 2.6 For a graph $G \cong P_n, n \geq 2$, the vertex-to-edge diameter $d_{1G} = n - 2$.

Lemma 2.7 is similar to Proposition 1.6.

Lemma 2.7 *For each $u \in V(G)$ of a connected graph G, $\{N_j^M[u] : N_j^M[u] \neq \emptyset, 0 \leq j \leq d_{1G}\}$ gives a partition of M.*

Proof If possible, let $e \in N_j^M[u] \cap N_k^M[u]$, for some $e \in M$ and $u \in V(G)$. Then $d(u,e) = j$ and $d(u,e) = k$, and hence $j = k$. Therefore, $N_j^M[u] \cap N_k^M[u] = \phi$ for any (j,k) with $j \neq k$. Now, clearly, $\bigcup_{j=o}^{d_{1G}} N_j^M[u] \subseteq M$. Also, for any $e \in M$, since G is connected, $d(u,e) = k$, for some $k \in \{0, 1, 2, \cdots, d_{1G}\}$. That is, $e \in N_k^M[u]$ for some $k \in \{0, 1, 2, \cdots, d_{1G}\}$ which implies $M \subseteq \bigcup_{j=o}^{d_{1G}} N_j^M[u]$. Hence $\bigcup_{j=o}^{d_{1G}} N_j^M[u] = M$. □

Corollary 2.8 *Each row of D_{1G}^M is a partition of $|M|$.*

Corollary 2.9 *Sum of the entries in each row of D_{1G}^M gives $|M|$ and sum of the entries in each row of D_{1G}^{*M} is less than or equal to $|M|$.*

§3. M-VTES-Distance Neighborhood Pattern Matrix of a VTES-dpd Graph

In this section we find out some results of D_{1G}^{*M} of a VTES-dpd-graph. From the definition of D_{1G}^{*M}, we have the following observations.

Observation 3.1 In any graph G, a nonempty $M \subseteq E(G)$ is a VTES-dpd-set if and only if no two rows of D_{1G}^{*M} are identical.

The following Theorem 3.2 shows that M is a proper subset of $E(G)$.

Theorem 3.2 *For a VTES-dpd-graph G of order p and size q, a VTES-dpd set M is such that $3 \leqq |M| \leqq q - 1$.*

Proof For lower bound, let M be a VTES-dpd set. If $M = \{e\}$, for some $e = uv \in E(G)$, then D_{1G}^{*M} contains a $2 \times (d_{1G} + 1)$ submatrix, such that the rows of submatrix represent the M-VTES-dnp of the vertices u and v in D_{1G}^{*M}. That is, entry 1 is at the first column of submatrix and the rows are as shown below,

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Hence D_{1G}^{*M} contains identical rows, Therefore M is not a VTES-dpd-set and hence $|M| \neq 1$.

Next, suppose $M = \{e_k, e_l\}$ for some $e_k = u_i u_j \in E(G)$ and $e_l = v_i v_j \in E(G)$. We consider the following cases.

Case 1. e_k is adjacent to e_l. Let $u_j = v_i$. Then $d_1(u_i, e_k) = d(v_j, e_l) = 0$ and $d(u_i, e_l) = d(v_j, e_k) = 1$. Then D_{1G}^{*M} contains a $2 \times (d_{1G} + 1)$ submatrix, such that the rows of submatrix represent the M-VTES-dnp of the vertices u_i and v_j in D_{1G}^{*M}. That is, entry 1 is at the first and second column of submatrix and rows are as shown below.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Case 2. e_k is not adjacent to e_l. Then the rows of the $2 \times (d_{1G} + 1)$ submatrix corresponding to the M-VTES-dnp of u_i and v_j in D_{1G}^{*M} are as follows.

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Thus D_{1G}^{*M} contains identical rows if $|M| = 2$ and so, M is not a VTES-dpd-set. Hence the lower bound follows.

For upper bound, suppose on contrary, there exist a VTES-dpd-set M with $|M| = q$. We prove by induction on $p \geq 2$.

Suppose $p = 2$ with $|M| = q$. Then the graph $G \cong K_2$ and $|M| = 1$. By lower bound $|M| \geq 3$, a contradiction. Suppose $p = 3$ with $|M| = q$ then the graph G is either $K_{1,2}$ or K_3. If $G \cong K_{1,2}$, with $|M| = q = 2$. By lower bound $|M| \geq 3$, a contradiction. If $G \cong K_3$ with $|M| = q = 3$. Then, we have a VTES-dpd-matrix

$$D_{1G}^{*M} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Clearly rows are identical hence, a contradiction. Therefore $|M| \neq q$ for $p = 2$ and $p = 3$. Suppose that $|M| \neq q$ for $p = n$. We claim that $|M| \neq q$ for $p = n + 1$. Let $V(G) = \{v_1, v_2, \cdots, v_n, v_{n+1}\}$ be the vertex set of G. One can observe that every graph has atleast one vertex-to-edge central vertex. Let $C_1(G)$ be set of vertex-to-edge central vertices. We consider the following cases.

Case 1. $|C_1(G)| = 1$. Let $v_i \in C_1(G)$, then D_{1G}^{*M} contains a $(degv_i) \times (d_{1G} + 1)$ submatrix, rows of which represent the M-VTES-dnp of the vertices $v_j \in N(v_i)$; $j = 1, 2, \cdots, degv_i$ in D_{1G}^{*M} as shown below.

$$\begin{pmatrix} 1 & 1 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Hence D_{1G}^{*M} contains identical rows , a contradiction. Hence $|M| \neq q$ for $p = n+1$. By mathematical induction, the result follows for all p.

Case 2. $|C_1(G)| \geq 2$. Let $C_1(G) = \{v_1, v_2, \cdots, v_i\}, i \geq 2$. For every $v_i, v_j \in C_1(G)$, there exists an edge e_k such that $d(v_i, e_k) = d(v_j, e_k)$. Then D_{1G}^{*M} contains a $2 \times (d_{1G}+1)$ submatrix, the rows of which represent the M-VTES-dnp of vertices v_i and v_j in D_{1G}^{*M} as shown below.

$$\begin{pmatrix} 1 & 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Hence D_{1G}^{*M} contains identical rows, a contradiction. Thus, $|M| \leqq q-1$. □

Lemma 3.3 *If $d_{1G} \leq 2$, then G does not possess a VTES-dpd-set.*

Proof One can verify from [6] Appendix 1,Table A_1. □

Corollary 3.4 *If $G \cong K_n$, $K_n - e$ or $K_{m,n}$, then G does not possess a VTES-dpd-set.*

Theorem 3.5 *A graph $G \cong P_n$ of order n admits a VTES-dpd-set if and only if $n \geq 5$.*

Proof Suppose that $G \cong P_n$, where $P_n = (v_1, e_1, v_2, e_2, v_3, e_3, \cdots, e_{n-1}, v_n,)$. Let $M = \{e_1, e_2, e_4\}$. Then

$$D_{1G}^{*M} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Now, we can partition D_{1G}^{*M} into two submatrices say, A and B where A is a $4 \times (d_{1G}+1)$ submatrix of the form

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Again we can find the 4×4 submatrix A_1 of A which is of the form

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

The remaining $4 \times (d_{1G} - 3)$ submatrix A_2 of A has all the entries as zero.

Also, Each i^th row, $1 \leq i \leq (n-4)$, of the submatrix B of order $(n-4) \times (d_{1G}+1)$ has entry 1 only in the i^{th}, $(i+2)^{nd}$,and $(i+3)^{rd}$ columns. None of the rows of the submatrices A and B are identical and hence no two rows of D_{1G}^{*M} are identical. Hence $\{e_1, e_2, e_4\}$ form a VTES-dpd-set. Therefore, any graph $G \cong P_n$ of order $n \geq 5$ admits a VTES-dpd-set.

Now to complete the proof we need to show that P_n is not a VTES-dpd-graph for $n \leq 4$. So, suppose that $G \cong P_n$ and $n \leq 4$. Since $n \leq 4$, $d_1(P_n) \leq 2$ for $n \leqq 4$. By Lemma 3.3, the proof follows. □

Corollary 3.6 *$G \cong P_5$ is the smallest VTES-dpd-graph with $M = \{e_1, e_2, e_4\}$*

Theorem 3.7 *A cycle $G \cong C_n$ of order n admits a VTES-dpd-set if and only if $n \geq 7$.*

Proof Let $C_n = (v_1, e_1, v_2, e_2, v_3, e_3, \cdots, e_{n-1}, v_n, e_1, v_1)$ be a cycle on n vertices. We consider the following cases.

Case 1. n is an even integer, and ≥ 8 . Let $M = \{e_1, e_2, e_4\}$. Then

$$D_{1G}^{*M} = \begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 1 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 1 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

Now, we can partition D_{1G}^{*M} into four submatrices say, A,B,C and D where A is a $4 \times (d_{1G}+1)$ submatrix of the form

$$\begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

Again we can find the 4×4 sub-matrix A_1 in A which is of the form

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Here the remaining $4 \times (d_{1G} - 3)$ sub-matrix A_2 of A has all the entries as zero. The submatrix B of order $\frac{(n-6)}{2} \times (d_{1G} + 1)$ is of the form

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Each i^th row, $1 \leq i \leq \frac{(n-6)}{2}$, of the submatrix B of order $(n-6) \times (d_{1G} + 1)$ has entry 1 only in the i^{th}, $(i+2)^{nd}$,and $(i+3)^{rd}$ columns.

We also choose submatrix C of order

$$(n - 4 - \frac{(n-6)}{2} - \frac{(n-8)}{2}) \times (d_{1G} + 1)$$

of the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Finally we can choose a submatrix D of order

$$\frac{(n-8)}{2} \times (d_{1G} + 1)$$

of the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 1 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly we can observe that none of the rows of submatrices A,B,C and D are identical and hence no two rows of D_{1G}^{*M} are identical. Therefore, any graph $G \cong C_n$ of order $n \geq 8$ admits a VTES-dpd-set.

Case 2. n is an odd integer and ≥ 7 Let $M = \{e_1, e_2, e_4\}$. Then

$$D_{1G}^{*M}=\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 1 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, we can partition D_{1G}^{*M} into four submatrices say, A,B,C and D, where A is a $4 \times (d_{1G}+1)$ submatrix of the form

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Again we can find the 4×4 submatrix A_1 of A which is of the form

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Here the remaining $4 \times (d_{1G}-3)$ submatrix A_2 of A has all the entries as zero. The submatrix B of order $\dfrac{(n-5)}{2} \times (d_{1G}+1)$ is of the form

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Each i^th row, $1 \leq i \leq \dfrac{(n-5)}{2}$, of the submatrix B of order $(n-5) \times (d_{1G}+1)$ has entry 1 only in the i^{th}, $(i+2)^{nd}$,and $(i+3)^{rd}$ columns.

Also, we choose a submatrix C of order $(n-4-\dfrac{(n-5)}{2}-\dfrac{n-7}{2}) \times (d_{1G}+1)$ of the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

Finally, we can choose a $(\dfrac{n-7}{2}) \times (d_{1G}+1)$ submatrix D of the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 1 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly, one can observe that the rows of A,B,C and D of D_{1G}^{*M} are not identical. Therefore, any graph $G \cong C_n$ of order $n \geq 7$ admits a VTES-dpd-set.

Now to complete the proof we need to show that the C_n is not a VTES-dpd-graph for $n \leq 6$. So, suppose that $G \cong C_n$ and $n \leq 6$. Then $d_{1G} \leq 2$. The proof follows by Lemma 3.3. □

Acknowledgments

The authors thank B.D.Acharya for his valuable suggestions during group discussion on 17^{th} June 2011. This research work is supported by DST Funded MRP.No–SB/EMEQ-119/2013.

References

[1] B.D.Acharya, Group discussion held in Mangalore University, India, on 17^{th} June 2011.

[2] S.C.Basak, D.Mills and B.D.Gute, Predicting bioactivity and toxicity of chemicals from mathematical descriptors: A chemical-cum-biochemical approach, In D.J.Klein and D.Brandas, editors, *Advances in Quantum Chemistry:Chemical Graph Theory: Wherefrom,wherefor and whereto?* Elsevier-Academic Press, 1-91, 2007.

[3] F.Buckley and F.Harary, *Distance in Graphs*, Addison Wesley Publishing Company, Advanced Book Programme, Redwood City, CA, 1990.

[4] K.A.Germina, Alphy Joseph and Sona Jose, Distance neighborhood pattern matrices, *Eur. J. Pure Appl. Math.*, Vol.3, No.4, 2010, 748-764.

[5] F.Harary and Melter, On the metric dimension of a graph, *Ars Combin.*, 2,191-195, 1976.

[6] F.Harary, *Graph Theory*, Addison Wesley Publ. Comp., Reading, Massachusetts, 1969.

[7] Kishori P.Narayankar, Lokesh S.B. Edge-Distance Pattern Distingushing Graph, Submitted.

[8] D.H.Rouvrey, Predicting chemistry from topology, *Scientific American*, 254, 9, 40-47,1986.

[9] A.P.Santhakumaran, Center of a graph with respect to edge, *SCIENTIA Series A: Mathematical Sciences*, Vol. 19 (2010), 13-23.

Math.Combin.Book Ser. Vol.3(2015), 116-124

Extended Results on Complementary Tree Domination Number and Chromatic Number of Graphs

S.Muthammai

(Government Arts College for Women (Autonomous), Pudukkottai - 622 001, India)

P.Vidhya

(S.D.N.B.Vaishnav College for Women (Autonomous), Chennai - 600 044, India)

E-mail: muthammai.sivakami@gmail.com, vidhya_lec@yahoo.co.in

Abstract: For any graph $G = (V, E)$ a subset $D \subseteq V$ is a dominating set if every vertex in $V - D$ is adjacent to at least one vertex in D. A dominating set is said to be a complementary tree dominating set if the induced subgraph $< V - D >$ is a tree. The minimum cardinality of a complementary tree dominating set is called the complementary tree domination number and is denoted by $\gamma_{ctd}(G)$. In this paper, we find an upper bound for $\gamma_{ctd}(G)+\chi(G) = 2p-5$ and $\gamma_{ctd}(G) + \chi(G) = 2p - 6$, p is the number of vertices in G.

Key Words: Domination number, complementary tree domination.

AMS(2010): 05C69.

§1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For graph theoretical terms, we refer Harary [1] and for terms related to domination we refer Haynes et al. [2].

A subset D of V is said to be a dominating set in G if every vertex in $V - D$ is adjacent to at least one vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. The concept of complementary tree domination was introduced by Muthammai, Bhanumathi and Vidhya [3]. A dominating set D is called a complementary tree domination set if the induced subgraph $< V - D >$ is a tree. The minimum cardinality of a complementary tree dominating set is called the complementary tree domination number of G, denoted by $\gamma_{ctd}(G)$ and such a set D is called a γ_{ctd} set. The minimum number of colours required to colour all the vertices such that adjacent vertices do not receive the same colour is the chromatic number $\chi(G)$.

In this paper, we obtain sharp upper bound for $\gamma_{ctd}(G)+\chi(G) = 2p-5$ and $\gamma_{ctd}(G)+\chi(G) = 2p-6$. We use the following previous results.

Theorem 1.1([1]) *For any connected graph G, $\chi(G) \leq \Delta(G) + 1$.*

Theorem 1.2([3]) *For any connected graph G with $p \geq 2$, $\gamma_{ctd}(G) \leq p - 1$.*

[1]Received December 29, 2014, Accepted August 30, 2015.

Theorem 1.3([3]) *Let G be a connected graph with $p \geq 2$. $\gamma_{ctd}(G) = p - 1$ if and only if G is a star on p vertices.*

Theorem 1.4([3]) *Let G be a connected graph containing a cycle. Then $\gamma_{ctd}(G) = p - 2$ if and only if G is isormorphic to one of the following graphs. C_p, K_p or G is the graph obtained by attaching pendant edges at at least one of the vertices of a complete graph.*

Theorem 1.5([3]) *Let T be a tree with p vertices which is not a star. Then $\gamma_{ctd}(T) = p - 2$ if and only if T is a path or T is obtained by attaching pendant edges at at least one of the end vertices.*

Theorem 1.6([4]) *For any connected graph G, $\gamma_{ctd}(G) + \chi(G) \leq 2p - 1$, $(p \geq 2)$. The equality holds if and only if $G \cong K_2$.*

Theorem 1.7([4]) *For any connected graph G, $\gamma_{ctd}(G) + \chi(G) = 2p - 2$ $(p \geq 3)$ if and only if $G \cong P_3$ or K_p, $p \geq 4$.*

Theorem 1.8([4]) *For any connected graph G, $\gamma_{ctd}(G) + \chi(G) = 2p - 3$ $(p \geq 4)$ if and only if G is a star on four vertices or G is the graph obtained by adding a pendant edge at exactly one vertex of K_{p-1}.*

Theorem 1.9([4]) *For any connected graph G, on p vertices, $\gamma_{ctd}(G) + \chi(G) = 2p - 4$ $(p \geq 5)$ if and only if G is one of the following graphs.*

(1) *G is a star on 5 vertices;*

(2) *G is a cycle on 4 (or) 5 vertices;*

(3) *G is the graph obtained by attaching exactly two pendant edges at one vertex or two vertices of K_{p-2};*

(4) *is the graph obtained by joining a new vertex to j $(2 \leq j \leq p - 2)$ vertices of K_{p-1}.*

§2. Main Results

Notation 2.1 The following notations are used in this paper:

(1) $K_n(p - n)$ is the set of graphs on n vertices obtained from K_n by attaching $(p - n)$, $(p > n)$ pendant edges at the vertices of K_n.

(2) $K_n(P_m)$ is the graph obtained from K_n by attaching a pendant edge of P_m to any one vertex of K_n.

(3) $K_n'(H)$ is the set of graphs obtained from K_n by joining each of the vertices of the graph H to the same i $(1 \leq i \leq n - 1)$ vertices of K_n.

(4) $K_n''(H)$ is the set of graphs obtained from K_n by joining each of the vertices of the graph H to distinct $(n - 1)$ vertices of K_n.

(5) $K_n'''(H)$ is the set of graphs obtained from K_n by joining all the vertices of H, each is adjacent to at least i $(2 \leq i \leq n - 1)$ vertices of K_n.

(6) $F_1(K_n, 2K_1)$ is the set of graphs obtained from K_n by joining one vertex of $2K_1$ to i $(2 \leq i \leq n - 1)$ vertices of K_n and the other vertex to any one vertex of K_n.

(7) $F_{21}(K_n, K_2)$ is the set of graphs obtained from K_n by joining one vertex of K_2 to i $(1 \leq i \leq n - 1)$ vertices of K_n.

(8) $F_{22}(K_n, K_2)$ is the set of graphs obtained from K_n by joining each of the vertices of K_2 to i $(1 \leq i \leq n - 1)$ distinct vertices of K_n.

(9) $F_3(K_n, 3K_1)$ is the set of graphs obtained from K_n by joining one vertex of $3K_1$ to any of the i $(1 \leq i \leq n-1)$ vertices of K_n and each of other two vertices of $3K_1$ to exactly one vertex of K_n.

(10) $F_{41}(K_n, K_2 \cup K_1)$ is the set of graphs obtained from K_n by joining one vertex of K_2 and the vertex of K_1 to distinct $(n-1)$ vertices of K_n.

(11) $F_{42}(K_n, K_2 \cup K_1)$ is the set of graphs obtained from K_n by joining one vertex of K_2 to i $(1 \leq i \leq n-1)$ vertices of K_n and the vertex of K_1 to any one vertex of K_n.

(12) $F_{43}(K_n, K_2 \cup K_1)$ is the set of graphs obtained from K_n by joining each of the vertices of $K_2 \cup K_1$ to vertices of K_n such that each vertex of $K_2 \cup K_1$ is adjacent to exactly one vertex of K_n.

(13) $F_{51}(K_n, P_3)$ is the set of graphs obtained from K_n by joining the central vertex of P_3 to i $(1 \leq i \leq n-1)$ vertices of K_n.

(14) $F_{52}(K_n, P_3)$ is the set of graphs obtained from K_n by joining a pendant vertex and the central vertex of P_3 to the same i $(1 \leq i \leq n-1)$ vertices of K_n.

(15) $F_{53}(K_n, P_3)$ is the set of graphs obtained from K_n by joining a pendant vertex and the central vertex of P_3 to distinct $(n-1)$ vertices of K_n.

In the following $\gamma_{ctd}(G) + \chi(G) = 2p-5$ and $\gamma_{ctd}(G) + \chi(G) = 2p-6$ are found.

Theorem 2.1 *Let G be a connected graph with p $(p \geq 6)$ vertices then $\gamma_{ctd}(G) + \chi(G) = 2p-5$ if and only if G is one of the following graphs:*

(a) *G is a star (or) a cycle on 6 vertices;*
(b) $G \in K_{p-3}(3)$*;*
(c) $G \in K'_{p-2}(K_2)$*;*
(d) $G \in F_1(K_{p-2}, 2K_1)$*;*
(e) $G \in F_{21}(K_{p-2}, K_2)$.

Proof If G is one of the graphs given in the theorem, then $\gamma_{ctd}(G) + \chi(G) = 2p-5$. Conversely, assume $\gamma_{ctd}(G) + \chi(G) = 2p-5$. This is possible only if

(*i*) $\gamma_{ctd}(G) = p-1$ and $\chi(G) = p-4$;
(*ii*) $\gamma_{ctd}(G) = p-2$ and $\chi(G) = p-3$;
(*iii*) $\gamma_{ctd}(G) = p-3$ and $\chi(G) = p-2$;
(*iv*) $\gamma_{ctd}(G) = p-4$ and $\chi(G) = p-1$;
(*v*) $\gamma_{ctd}(G) = p-5$ and $\chi(G) = p$.

Case 1. $\gamma_{ctd}(G) = p-1$ and $\chi(G) = p-4$.

But, $\gamma_{ctd}(G) = p-1$ if and only if G is star $K_{1,p-1}$ on p vertices (Theorem 1.3, [3]). For a star G, $\chi(G) = 2$. Therefore, $\chi(G) = p-4$ implies that $p = 6$ that is, G is a star on 6 vertices.

Case 2. $\gamma_{ctd}(G) = p-2$ and $\chi(G) = p-3$.

But, $\gamma_{ctd}(G) = p-2$ implies that G is one of the following graphs (a) C_p, cycle on p vertices (b) K_p, complete graph on p vertices (c) G is the graph obtained by attaching pendant edges at least one of the vertices of a complete graph (d) G is a path (e) G is obtained from a path of at least three vertices, by attaching pendant edges at at least one of the end vertices of the path.

G cannot be one of the graphs mentioned in (b), (d) and (e), since if $G \cong K_p$, then $\chi(G) = p$ and if G is a path (or) as in (e), then $\chi(G) = 2$ and hence $p = 5$.

If $G \cong C_p$ then $\chi(G) = p-3$ implies $p = 5$ (or) 6. But, G has at least 6 vertices and hence $G \cong C_6$. Let G be a graph obtained by attaching pendant edge at at least one of the vertices of a complete graph.

But $\chi(G) = p-3$ implies that, G is the graph on p vertices obtained from K_{p-3} by attaching three pendant edges.
That is, $G \in K_{p-3}(3)$.

Case 3. $\gamma_{ctd}(G) = p-3$ and $\chi(G) = p-2$.

$\chi(G) = p-2$ implies that either G contains or does not contain a clique K_{p-2} on $(p-2)$ vertices. Assume G contains a clique K_{p-2} on $(p-2)$ vertices. Let $V(K_{p-2}) = \{u_1, u_2, \cdots, u_{p-2}\}$ and $D = V(G) - V(K_{p-2}) = \{x, y\}$.

Since G is connected, at least one of x and y is adjacent to vertices of K_{p-2}. Also both x and y are adjacent to at most $(p-3)$ vertices of K_{p-2}.

Subcase 3.1. x and y are non adjacent.

If both x and y are adjacent to same u_i $(1 \leq i \leq p-2)$ then $V - D = V(G) - \{$any two vertices of $K_{p-2}\}$ forms a minimum ctd-set of G, since the pendant vertices x and y must be in any ctd-set and hence $\gamma_{ctd}(G) = p-2$.

Similarly, if both x and y are adjacent to same i $(2 \leq i \leq p-3)$ vertices of K_{p-2}, then the set $V(G) - \{x, y, u_i, u_j\}$ where $u_i \in N(x) \cap K_{p-2}$ and $u_j \in (N(x))^c \cap K_{p-2}$ forms a minimum ctd-set and hence $\gamma_{ctd}(G) = p-4$.

Let x be adjacent to at least i vertices of K_{p-2}, where $2 \leq i \leq p-3$. If y is adjacent to at least two vertices of K_{p-2}, then also $\gamma_{ctd}(G) = p-4$. Therefore, y is adjacent to exactly one vertex of K_{p-2}. That is, G is the graph obtained by joining two non-adjacent vertices to vertices of K_{p-2}, such that one vertex is adjacent to i $(2 \leq i \leq p-3)$ vertices and the other vertex is adjacent to exactly one vertex of K_{p-2}. That is, $G \in F_1(K_{p-2}, 2K_1)$.

Subcase 3.2. x and y are adjacent.

If $N(x) \cap K_{p-2}$ and $N(y) \cap K_{p-2}$ are distinct, then $\gamma_{ctd}(G) = p-4$, since the set $V(G) - \{x, y, u_i, u_j\}$, where $u_i \in N(x) \cap (N(y))^c \cap K_{p-2}$ and $u_j \in (N(x))^c \cap (N(y))^c \cap K_{p-2}$ forms a minimum ctd-set. Therefore, $N(x) \cap K_{p-2}$ and $N(y) \cap K_{p-2}$ are equal. Hence, G is the graph obtained from K_{p-2} by joining the two vertices of K_2 to the same i $(1 \leq i \leq p-3)$ vertices of K_{p-2} (or) G is the graph obtained from K_{p-2} by joining one vertex of K_2 to i $(1 \leq i \leq p-3)$ vertices of K_{p-2}. Therefore, $G \in K'_{p-2}(K_2)$ (or) $G \in F_{21}(K_{p-2}, K_2)$.

If G does not contain a clique on $(p-2)$ vertices then it can be seen that no new graph exists.

Case 4. $\gamma_{ctd}(G) = p-4$ and $\chi(G) = p-1$.

$\chi(G) = p-1$ implies that either G contains or does not contain a clique K_{p-1} on $(p-1)$ vertices. Assume G contains a clique K_{p-1} on $(p-1)$ vertices. Let $V(G) - V(K_{p-1}) = \{x\}$. Since G is connected, x is adjacent to at least one of the vertices of K_{p-1}. Also, x is not adjacent to all the vertices of K_{p-2}, since otherwise $G \cong K_p$. Then either $V(G) - \{u_i, u_j\}$ (or) $V(G) - \{x, u_i, u_j\}$, where $u_i \in N(x) \cap K_{p-1}$ and $u_j \in (N(x))^c \cap K_{p-1}$ forms a minimum ctd-set. Hence in this case, no graph exists. If G does not contain a clique K_{p-1} on $(p-1)$ vertices.

Case 5. $\gamma_{ctd}(G) = p-5$ and $\chi(G) = p$.

$\chi(G) = p$ implies $G \cong K_p$. But, $\gamma_{ctd}(K_p) = p-2$. Here also, no graph exists. From cases 1 - 5, G can be one of the following graphs:

(a) G is a star (or) a cycle on 6 vertices;

(b) $G \in K_{p-3}(3)$;

(c) $G \in K'_{p-2}(K_2)$;
(d) $G \in F_1(K_{p-2}, 2K_1)$;
(e) $G \in F_{21}(K_{p-2}, K_2)$. □

Remark 2.1 For any connected graph with p $(3 \le p \le 5)$ vertices, $\gamma_{ctd}(G) + \chi(G) = 2p - 5$ if and only if G is one of the following graphs.

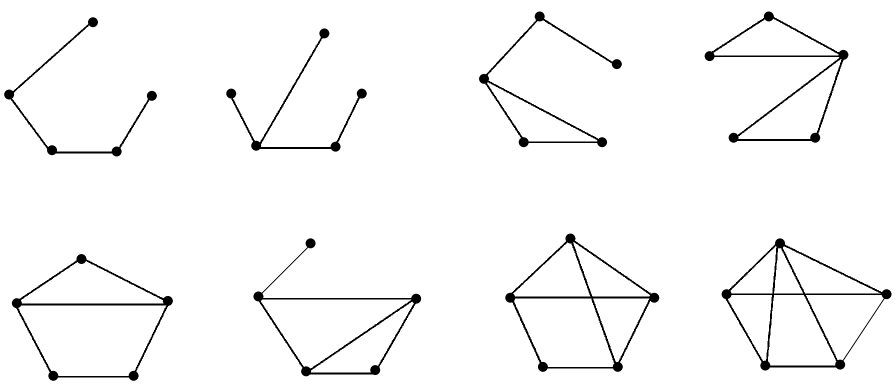


Fig.1

Theorem 2.2 *For any connected graph G with p $(p \ge 7)$ vertices, $\gamma_{ctd}(G) + \chi(G) = 2p - 6$ if and only if G is one of the following graphs:*

(a) *G is a star (or) a cycle on 7 vertices;*
(b) $G \in K_{p-4}(4)$;
(c) $G \in F_3(K_{p-3}, 3K_1)$;
(d) $G \in K'_{p-3}(K_3)$;
(e) $G \in K''_{p-3}(K_3)$;
(f) $G \in F_{41}(K_{p-3}, K_2 \cup K_1)$;
(g) $G \in F_{42}(K_{p-3}, K_2 \cup K_1)$;
(h) $G \in F_{43}(K_{p-3}, K_2 \cup K_1)$;
(i) $G \in K_{p-3}(P_4)$;
(j) $G \in F_{51}(K_{p-3}, P_3)$;
(k) $G \in F_{52}(K_{p-3}, P_3)$;
(l) $G \in F_{53}(K_{p-3}, P_3)$;
(m) $G \in K''_{p-3}(P_3)$;
(n) $G \in K'''_{p-2}(2K_1)$;
(o) $G \in F_{22}(K_{p-2}, K_2)$.

Proof If G is one of the graphs given in the theorem, then $\gamma_{ctd}(G) + \chi(G) = 2p - 6$. Conversely, assume $\gamma_{ctd}(G) + \chi(G) = 2p - 6$. This possible, only if

(*i*) $\gamma_{ctd}(G) = p - 1$ and $\chi(G) = p - 5$;
(*ii*) $\gamma_{ctd}(G) = p - 2$ and $\chi(G) = p - 4$;
(*iii*) $\gamma_{ctd}(G) = p - 3$ and $\chi(G) = p - 3$;
(*iv*) $\gamma_{ctd}(G) = p - 4$ and $\chi(G) = p - 2$;
(*v*) $\gamma_{ctd}(G) = p - 5$ and $\chi(G) = p - 1$;

(vi) $\gamma_{ctd}(G) = p - 6$ and $\chi(G) = p$.

Case 1. $\gamma_{ctd}(G) = p - 1$ and $\chi(G) = p - 5$

But, $\gamma_{ctd}(G) = p - 1$ if and only if G is a star $K_{1,p-1}$ on p vertices. But, for a star $\chi(G) = 2$. Hence, $p = 7$. That is, G is a star on 7 vertices.

Case 2. $\gamma_{ctd}(G) = p - 2$ and $\chi(G) = p - 4$

But, $\gamma_{ctd}(G) = p - 2$ if and only if

(a) $G \cong C_p$;

(b) $G \cong K_p$;

(c) G is the graph obtained by attaching pendant edges at at least one of the vertices of a complete graph;

(d) G is a path;

(e) G is obtained from of path of at least three vertices by attaching pendant edges at at least one of the end vertices of the path.

As in case 2 of Theorem 2.1.

G is a cycle on 7 vertices (or) G is the graph on p vertices obtained from K_{p-4} by attaching four pendant edges. That is, $G \cong C_7$ (or) $K_{p-4}(4)$.

Case 3. $\gamma_{ctd}(G) = \chi(G) = p - 3$

$\chi(G) = p - 3$ implies that either G contains or does not contains a clique K_{p-3} on $(p - 3)$ vertices. Assume G contains a clique K_{p-3} on $(p - 3)$ vertices. Let $V(K_{p-3}) = \{u_1, u_2, \cdots, u_{p-3}\}$ and $D = V(G) - V(K_{p-3}) = \{x, y, z\}$. Each of x, y, z is not adjacent to all the vertices of K_{p-3}. $< D >= \overline{K_3}, K_3, P_3$ (or) $K_2 \cup K_1$.

Subcase 3.1. $< D >\cong \overline{K_3}$.

Since G is connected, every vertex of D is adjacent to at least one vertex of K_{p-3}. Let x be adjacent to i $(1 \leq i \leq p - 4)$ vertices of K_{p-3}.

If y (or) z is adjacent to at least two vertices of K_{p-3}, then $\gamma_{ctd}(G) \leq p - 4$. Therefore, both y and z are adjacent to exactly one vertex of K_{p-3}. That is, G is the graph obtained from K_{p-3} by joining vertices of $3K_1$ to the vertices of K_{p-3} such that one is adjacent to any of the i $(1 \leq i \leq p-4)$ vertices of K_{p-3} and each of the remaining two is adjacent to exactly one vertex of K_{p-3} and hence $G \in F_3(K_{p-3}, 3K_1)$.

Subcase 3.2. $< D >\cong K_3$.

Since G is connected, at least one vertex of K_3 is adjacent to vertices of K_{p-3}. If there exist vertices $u_i, u_j \in K_{p-3}$ such that $u_i \in N(x) \cap (N(y))^c \cap K_{p-3}$ and $u_j \in (N(x))^c \cap (N(y))^c \cap K_{p-3}$, then the set $V(G) - \{x, y, u_i, u_j\}$ is a γ_{ctd}-set of G and hence $\gamma_{ctd}(G) = p - 4$.

Similarly in the case, when $u_i \in N(y) \cap (N(z))^c$ and $u_j \in (N(x))^c \cap (N(y))^c$ (or) $u_i \in N(z) \cap (N(x))^c$ and $u_j \in (N(z))^c \cap (N(x))^c$ in K_{p-3}.

Therefore, either (i) $N(x) \cap K_{p-3} = N(y) \cap K_{p-3} = N(z) \cap K_{p-3}$ (or) (ii) $N(x) \cap K_{p-3}$, $N(y) \cap K_{p-3}$, $N(z) \cap K_{p-3}$ are mutually distinct and each has $(p-4)$ vertices. That is, G is the graph obtained from K_{p-3} by joining each of the vertices of K_3 either to the same i $(1 \leq i \leq p-4)$ vertices of K_{p-3} (or) to distinct $(p-4)$ vertices of K_{p-3}. Therefore, $G \in K'_{p-3}(K_3)$ (or) $G \in K''_{p-3}(K_3)$.

Subcase 3.3. $< D >\cong K_2 \cup K_1$.

Let $x, y \in V(K_2)$ and $z \in V(K_1)$ since G is connected, at least one of the vertices of K_2 and z is adjacent to vertices of K_{p-3}. Denote $G \cap K_{p-3}$ by G_1.

(i) Let one of x and y, say x be adjacent to vertices of K_{p-3}. That is, $deg_{G_1}(y) = 1$.

Let x be adjacent to at least two vertices of K_{p-3}. That is, $deg_{G_1}(x) \geq 2$. Assume $deg_{G_1}(z) \geq 2$. If there exist $u_i, u_j \in K_{p-3}$ such that $u_i \in N(x) \cap N(z)$ and $u_j \in (N(x))^c \cap (N(z))^c$ or if $N(x) \cap K_{p-3} = N(z) \cap K_{p-3}$ and if each set has $(p-4)$ vertices, then $\gamma_{ctd}(G) = p-4$. Therefore, we have the following cases:

(a) $N(x) \cap K_{p-3}$ and $N(z) \cap K_{p-3}$ are distinct, and each set has $(p-4)$ vertices or

(b) $deg_{G_1}(z) = 1$. That is, G is the graph obtained from K_{p-3} by joining exactly one of the vertices of K_2 and a new vertex to distinct $(p-4)$ vertices of K_{p-3} or G is the graph obtained from K_{p-3} by attaching a pendant edge and joining exactly one vertex of K_2 to i $(1 \leq i \leq p-4)$ vertices of K_{p-3}. That is, $G \in F_{41}(K_{p-3}, K_2 \cup K_1)$ or $G \in F_{42}(K_{p-3}, K_2 \cup K_1)$.

(ii) If each of x, y, z is adjacent to at least two vertices of K_{p-3}, then either $V(G) - \{x, y, z, u_i, u_j\}$, where $u_i \in N(x) \cap (N(y))^c \cap (N(z))^c \cap K_{p-3}$ and $u_j \in N(z) \cap (N(x))^c \cap (N(y))^c \cap K_{p-3}$ (or) $V(G) - \{x, y, u_i, u_j\}$, where $u_i \in N(x) \cap (N(y))^c \cap K_{p-3}$ and $u_j \in (N(x))^c \cap (N(y))^c \cap K_{p-3}$ is a γ_{ctd}-set of G.

Similarly, if either $N(x) \cap K_{p-3} = N(y) \cap K_{p-3} = N(z) \cap K_{p-3}$ and $2 \leq |N(x) \cap K_{p-3}| \leq p-4$. (or) $N(x) \cap K_{p-3}$, $N(y) \cap K_{p-3}$, and $N(z) \cap K_{p-3}$ are distinct and each set has the same number i $(2 \leq i \leq p-4)$ of elements, then also $\gamma_{ctd}(G) = p-4$.

Hence, each of x, y and z is adjacent to exactly one vertex of K_{p-3}. That is, G is the graph obtained from K_{p-3} by attaching a pendant edge and joining two vertices of K_2 to vertices of K_{p-3} such that each is adjacent to exactly one vertex of K_{p-3}. Hence, $G \in F_{43}(K_{p-2}, K_2 \cup K_1)$.

Subcase 3.4. $< D > \cong P_3$.

Since G is connected, at least one of the vertices of P_3 is adjacent to vertices of K_{p-3}. Let x and z be the pendant vertices and y be the central vertex of P_3.

(i) Assume exactly one of x, y, z is adjacent to vertices of K_{p-3}. If $deg_{G_1}(x) \geq 2$, then $\gamma_{ctd}(G) = p-4$. Hence, $deg_{G_1}(x) = 1$. That is, G is the graph obtained from K_{p-3} by attaching a path of length 3 at a vertex of K_{p-3} (or) that is, $G \in K_{p-3}(P_4)$ (or) G is the graph obtained from K_{p-3} by joining the central vertex of P_3 to i $(1 \leq i \leq p-4)$ vertices of K_{p-3}, that is, $G \in F_{51}(K_{p-3}, P_3)$.

(ii) Assume any two of x, y, z are adjacent to vertices of K_{p-3}.

(a) If x and z are adjacent to vertices of K_{p-3}, then $\gamma_{ctd}(G) = p-4$.

(b) Let x and y be adjacent to vertices of K_{p-3}. If there exist vertices $u_i, u_j \in K_{p-3}$ such that $u_i \in N(x) \cap (N(y))^c$ and $u_j \in (N(x))^c \cap (N(y))^c$, then also $\gamma_{ctd}(G) = p-4$. Therefore, either

(a) $N(x) \cap K_{p-3} = N(y) \cap K_{p-3}$ or

(b) $N(x) \cap K_{p-3}$ and $N(y) \cap K_{p-3}$ are distinct and each set has $(p-4)$ vertices. That is, G is the graph obtained from K_{p-3} by joining one pendant vertex and the central vertex of P_3 to the same i $(1 \leq i \leq p-4)$ vertices of K_{p-3} (or) G is the graph obtained from K_{p-3} by joining one pendant vertex and the central vertex of P_3 to the distinct $(p-4)$ vertices of K_{p-3}. i.e., $G \in F_{52}(K_{p-3}, P_3)$ or $G \in F_{53}(K_{p-3}, P_3)$.

(iii) Assume x, y and z are adjacent to vertices of K_{p-3}. As in Subcase 3.3, if $N(x) \in K_{p-3} = N(y) \cap K_{p-3} = N(z) \cap K_{p-3}$ and $1 \leq |N(x) \cap K_{p-3}| \leq (p-4)$ or $N(x) \cap K_{p-3}$, $N(y) \cap K_{p-3}$ and $N(z) \cap K_{p-3}$ are distinct and each of these sets are distinct and has $(p-4)$ vertices. Hence, G is the graph obtained from K_{p-3} by joining each of the vertices of P_3 to distinct $(p-4)$ vertices of K_{p-3}.

That is, $G \in K''_{p-3}(P_3)$.

If G does not contain a clique K_{p-3} on $(p-3)$ vertices, then it can be verified that no new graph exists.

Case 4. $\gamma_{ctd}(G) = p - 4$ and $\chi(G) = p - 2$.

$\chi(G) = p-2$ implies that G either contains or does not contains a clique K_{p-2} on $(p-2)$ vertices. Assume G contains a clique K_{p-2} on $p-2$ vertices. Let $V(G) - V(K_{p-2}) = \{x, y\}$. If x and y are non-adjacent then as in Subcase 3.1 of Theorem 2.1, G is the graph obtained from K_{p-2} by joining two non-adjacent vertices to vertices of K_{p-2} such that each is adjacent to at least i $(2 \leq i \leq p-3)$ vertices of K_{p-2}. That is, $G \in K'''_{p-2}(2K_1)$.

If x and y are adjacent, then as in subcase 3.2 of Theorem 2.1, G is the graph obtained from K_{p-2} by joining each of the vertices of K_2 to i $(1 \leq i \leq p-3)$ distinct vertices of K_{p-2}. That is, $G \in F_{22}(K_{p-2}, K_2)$. If G does not contains a clique on $p-2$ vertices, then no new graph exists. For the cases $\gamma_{ctd}(G) = p-5$ and $\chi(G) = p-1$ and $\gamma_{ctd}(G) = p-6$ and $\chi(G) = p$, no graph exists.

From cases 1 - 4, we can conclude that G can be one of the graphs given in the theorem. □

Remark 2.2 For any connected graph with p $(4 \leq p \leq 6)$ vertices, $\gamma_{ctd}(G) + \chi(G) = 2p - 6$ if and only if G is one of the following graphs.

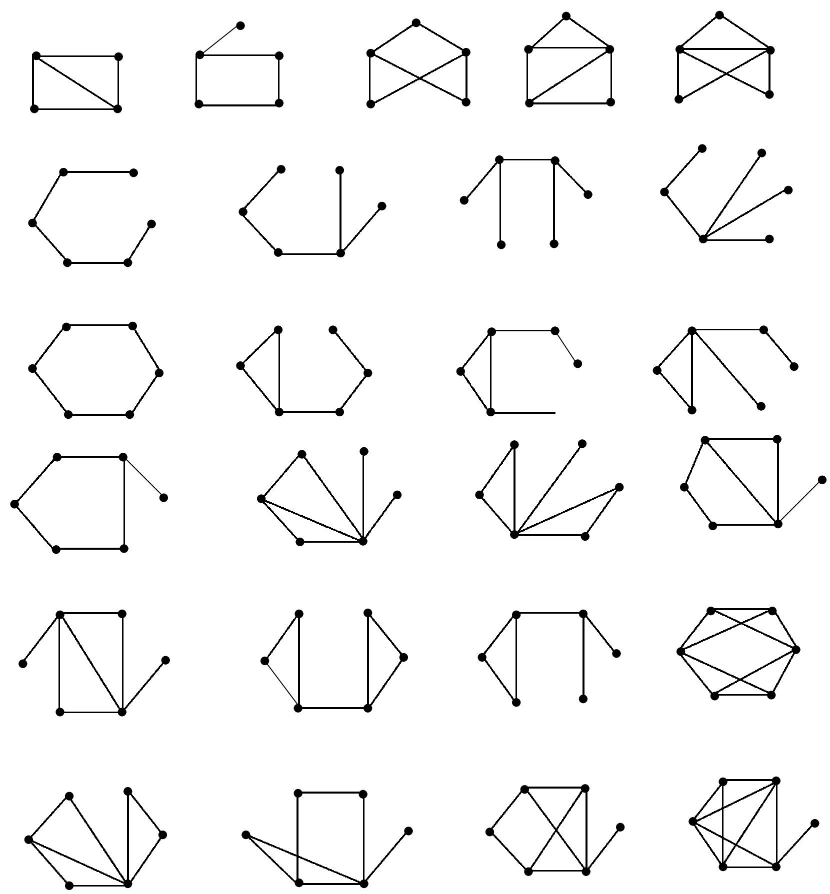


Fig.2

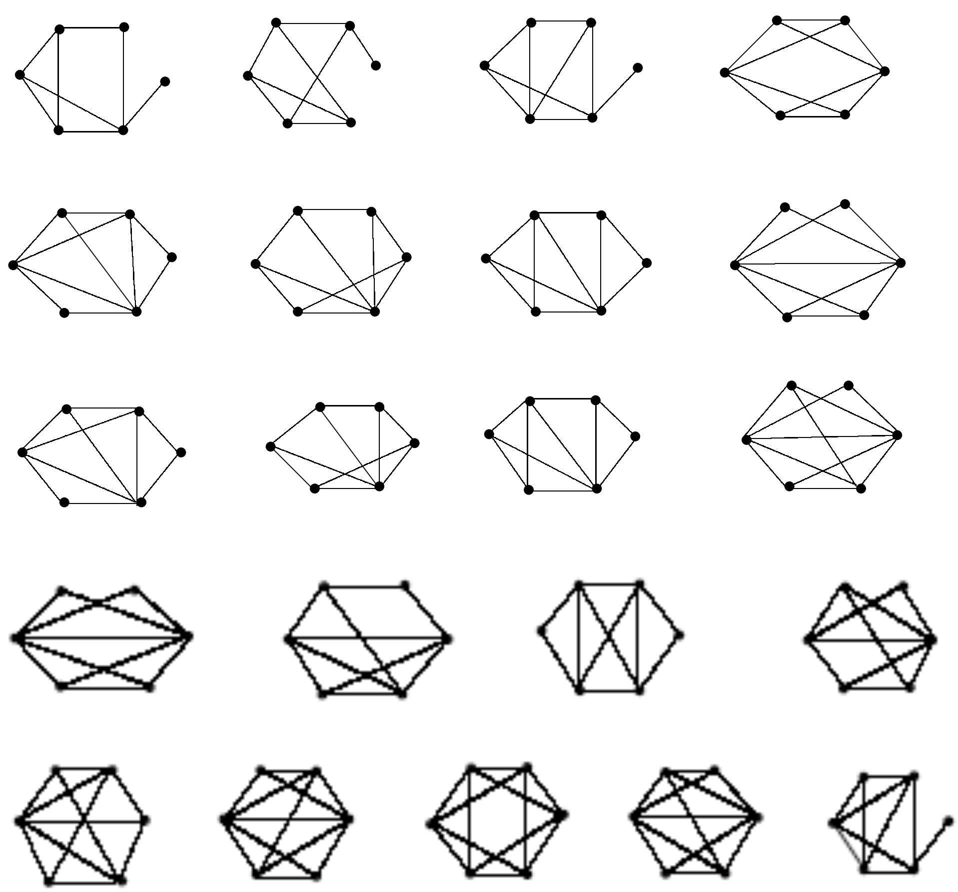


Fig.3

References

[1] F.Harary, *Graph Theory*, Addison Wesley, Reading Massachuretts, 1972.

[2] T.W.Haynes, S.T.Hedetniemi and P.J.Slater, *Fundamental of Domination in Graphs*, Marcel Dekker Inc., New York, 1998.

[3] S.Muthammai, M.Bhanumathi and P.Vidhya, Complementary tree domination number of a graph, *Int. Mathematical Forum*, Vol. 6(2011), No. 26, 1273-1282.

[4] S.Muthammai and P.Vidhya, Complementary tree domination number and chromatic number of graphs, *International Journal of Mathematics and Scientific Computing*, Vol.1, 1(2011), 66–68.

Math.Combin.Book Ser. Vol.3(2015), 125-133

On Integer Additive Set-Sequential Graphs

N.K.Sudev

Department of Mathematics

Vidya Academy of Science & Technology, Thalakkottukara, Thrissur - 680501, India

K.A.Germina

PG & Research Department of Mathematics

Mary Matha Arts & Science College, Mnanthavady, Wayanad-670645, India)

E-mail:sudevnk@gmail.com, srgerminaka@gmail.com

Abstract: A set-labeling of a graph G is an injective function $f : V(G) \to \mathcal{P}(X)$, where X is a finite set of non-negative integers and a set-indexer of G is a set-labeling such that the induced function $f^{\oplus} : E(G) \to \mathcal{P}(X) - \{\emptyset\}$ defined by $f^{\oplus}(uv) = f(u) \oplus f(v)$ for every $uv \in E(G)$ is also injective. A set-indexer $f : V(G) \to \mathcal{P}(X)$ is called a set-sequential labeling of G if $f^{\oplus}(V(G) \cup E(G)) = \mathcal{P}(X) - \{\emptyset\}$. A graph G which admits a set-sequential labeling is called a set-sequential graph. An integer additive set-labeling is an injective function $f : V(G) \to \mathcal{P}(\mathbb{N}_0)$, $\mathbb{N}_0$ is the set of all non-negative integers and an integer additive set-indexer is an integer additive set-labeling such that the induced function $f^+ : E(G) \to \mathcal{P}(\mathbb{N}_0)$ defined by $f^+(uv) = f(u) + f(v)$ is also injective. In this paper, we extend the concepts of set-sequential labeling to integer additive set-labelings of graphs and provide some results on them.

Key Words: Integer additive set-indexers, set-sequential graphs, integer additive set-labeling, integer additive set-sequential labeling, integer additive set-sequential graphs.

AMS(2010): 05C78.

§1. Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [4], [5] and [9] and for more about graph labeling, we refer to [6]. Unless mentioned otherwise, all graphs considered here are simple, finite and have no isolated vertices.

All sets mentioned in this paper are finite sets of non-negative integers. We denote the cardinality of a set A by $|A|$. We denote, by X, the finite ground set of non-negative integers that is used for set-labeling the elements of G and cardinality of X by n.

The research in graph labeling commenced with the introduction of β-valuations of graphs in [10]. Analogous to the number valuations of graphs, the concepts of set-labelings and set-indexers of graphs are introduced in [1] as follows.

Let G be a (p, q)-graph. Let X, Y and Z be non-empty sets and $\mathcal{P}(X)$, $\mathcal{P}(Y)$ and $\mathcal{P}(Z)$ be their power sets. Then, the functions $f : V(G) \to \mathcal{P}(X)$, $f : E(G) \to \mathcal{P}(Y)$ and $f : V(G) \cup E(G) \to \mathcal{P}(Z)$ are called the *set-assignments* of vertices, edges and elements of G respectively. By a set-assignment

[1]Received December 31, 2014, Accepted August 31, 2015.

of a graph, we mean any one of them. A set-assignment is called a *set-labeling* or a *set-valuation* if it is injective.

A graph with a set-labeling f is denoted by (G, f) and is referred to as a *set-labeled graph* or a *set-valued graph*. For a (p, q)- graph $G = (V, E)$ and a non-empty set X of cardinality n, a *set-indexer* of G is defined as an injective set-valued function $f : V(G) \to \mathcal{P}(X)$ such that the function $f^{\oplus} : E(G) \to \mathcal{P}(X) - \{\emptyset\}$ defined by $f^{\oplus}(uv) = f(u) \oplus f(v)$ for every $uv \in E(G)$ is also injective, where $\mathcal{P}(X)$ is the set of all subsets of X and $\oplus$ is the symmetric difference of sets.

Theorem 1.1([1]) *Every graph has a set-indexer.*

Analogous to graceful labeling of graphs, the concept of set-graceful labeling and set-sequential labeling of a graph are defined in [1] as follows.

Let G be a graph and let X be a non-empty set. A set-indexer $f : V(G) \to \mathcal{P}(X)$ is called a *set-graceful labeling* of G if $f^{\oplus}(E(G)) = \mathcal{P}(X) - \{\emptyset\}$. A graph G which admits a set-graceful labeling is called a *set-graceful graph*.

Let G be a graph and let X be a non-empty set. A set-indexer $f : V(G) \to \mathcal{P}(X)$ is called a *set-sequential labeling* of G if $f^{\oplus}(V(G) \cup E(G)) = \mathcal{P}(X) - \{\emptyset\}$. A graph G which admits a set-sequential labeling is called a *set-sequential graph*.

Let A and B be two non-empty sets. Then, their *sum set*, denoted by $A + B$, is defined to be the set $A + B = \{a + b : a \in A, b \in B\}$. If $C = A + B$, then A and B are said to be the *summands* of C. Using the concepts of sum sets of sets of non-negative integers, the notion of integer additive set-labeling of a given graph G is introduced as follows.

Let $\mathbb{N}_0$ be the set of all non-negative integers. An *integer additive set-labeling* (IASL, in short) of graph G is an injective function $f : V(G) \to \mathcal{P}(\mathbb{N}_0)$ such that the induced function $f^+ : E(G) \to \mathcal{P}(\mathbb{N}_0)$ is defined by $f^+(uv) = f(u) + f(v)$ for $\forall uv \in E(G)$. A graph G which admits an IASL is called an IASL graph.

An *integer additive set-labeling* f is an integer additive set-indexer (IASI, in short) if the induced function $f^+ : E(G) \to \mathcal{P}(\mathbb{N}_0)$ defined by $f^+(uv) = f(u) + f(v)$ is injective(see [7]). A graph G which admits an IASI is called an IASI graph.

The following notions are introduced in [11] and [8]. The cardinality of the set-label of an element (vertex or edge) of a graph G is called the *set-indexing number* of that element. An IASL (or an IASI) is said to be a k-uniform IASL (or k-uniform IASI) if $|f^+(e)| = k \ \forall \ e \in E(G)$. The vertex set $V(G)$ is called *l-uniformly set-indexed*, if all the vertices of G have the set-indexing number l.

Definition 1.2([13]) *Let G be a graph and let X be a non-empty set. An integer additive set-indexer $f : V(G) \to \mathcal{P}(X) - \{\emptyset\}$ is called a* integer additive set-graceful labeling *(IASGL, in short) of G if $f^+(E(G)) = \mathcal{P}(X) - \{\emptyset, \{0\}\}$. A graph G which admits an integer additive set-graceful labeling is called an* integer additive set-graceful graph *(in short, IASG-graph).*

Motivated from the studies made in [2] and [3], in this paper, we extend the concepts of set-sequential labelings of graphs to integer additive set-sequential labelings and establish some results on them.

§2. IASSL of Graphs

First, note that under an integer additive set-labeling, no element of a given graph can have $\emptyset$ as its

set-labeling. Hence, we need to consider only non-empty subsets of X for set-labeling the elements of G.

Let f be an integer additive set-indexer of a given graph G. Define a function $f^* : V(G) \cup E(G) \to \mathcal{P}(X) - \{\emptyset\}$ as follows.

$$f^*(x) = \begin{cases} f(x) & \text{if } x \in V(G) \\ f^+(x) & \text{if } x \in E(G) \end{cases} \tag{2.1}$$

Clearly, $f^*[V(G) \cup E(G)] = f(V(G)) \cup f^+(E(G))$. By the notation, $f^*(G)$, we mean $f^*[V(G) \cup E(G)]$. Then, f^* is an extension of both f and f^+ of G. Throughout our discussions in this paper, the function f^* is as per the definition in Equation 2.1.

Using the definition of new induced function f^* of f, we introduce the following notion as a sum set analogue of set-sequential graphs.

Definition 2.1 *An IASI f of G is said to be an* integer additive set-sequential labeling *(IASSL) if the induced function $f^*(G) = f(V(G)) \cup f^+E(G)) = \mathcal{P}(X) - \{\emptyset\}$. A graph G which admits an IASSL may be called an* integer additive set-sequential graph *(IASS-graph).*

Hence, an integer additive set-sequential indexer can be defined as follows.

Definition 2.2 *An integer additive set-sequential labeling f of a given graph G is said to be an integer additive set-sequential indexer (IASSI) if the induced function f^* is also injective. A graph G which admits an IASSI may be called an* integer additive set-sequential indexed graph *(IASSI-graph).*

A question that arouses much in this context is about the comparison between an IASGL and an IASSL of a given graph if they exist. The following theorem explains the relation between an IASGL and an IASSL of a given graph G.

Theorem 2.3 *Every integer additive set-graceful labeling of a graph G is also an integer additive set-sequential labeling of G.*

Proof Let f be an IASGL defined on a given graph G. Then, $\{0\} \in f(V(G))$ (see [13]) and $|f^+(E(G))| = \mathcal{P}(X) - \{\emptyset, \{0\}\}$. Then, $f^*(G)$ contains all non-empty subsets of X. Therefore, f is an IASSL of G. □

Let us now verify the injectivity of the function f^* in the following proposition.

Proposition 2.4 *Let G be a graph without isolated vertices. If the function f^* is an injective, then no vertex of G can have a set-label $\{0\}$.*

Proof If possible let a vertex, say v, has the set-label $\{0\}$. Since G is connected, v is adjacent to at least one vertex in G. Let u be an adjacent vertex of v in G and u has a set-label $A \subset X$. Then, $f^*(u) = f(u) = A$ and $f^*(uv) = f^+(uv) = A$, which is a contradiction to the hypothesis that f^* is injective. □

In view of Observation 2.4, we notice the following points.

Remark 2.5 Suppose that the function f^* defined in (2.1) is injective. Then, if one vertex v of G has the set label $\{0\}$, then v is an isolated vertex of G.

Remark 2.6 If the function f^* defined in (2.1) is injective, then no edge of G can also have the set

label $\{0\}$.

The following result is an immediate consequence of the addition theorem on sets in set theory and provides a relation connecting the size and order of a given IASS-graph G and the cardinality of its ground set X.

Proposition 2.7 *Let G be a graph on n vertices and m edges. If f is an IASSL of a graph G with respect to a ground set X, then $m+n = 2^{|X|} - (1+\kappa)$, where κ is the number of subsets of X which is the set-label of both a vertex and an edge.*

Proof Let f be an IASSL defined on a given graph G. Then, $|f^*(G)| = |f(V(G)) \cup f^+(E(G))| = |\mathcal{P}(X) - \{\emptyset\}| = 2^{|X|} - 1$. But by addition theorem on sets, we have

$$\begin{aligned} |f^*(G)| &= |f(V(G)) \cup f^+(E(G))| \\ \text{That is,}\quad 2^{|X|} - 1 &= |f(V(G))| + |f^+(E(G))| - |f(V(G)) \cap f^+(E(G))| \\ &= |V| + |E| - \kappa \\ \Longrightarrow \quad &= m + n - \kappa \\ \text{Whence}\quad m+n &= 2^{|X|} - 1 - \kappa. \end{aligned}$$

This completes the proof. □

We say that two sets A and B are of *same parity* if their cardinalities are simultaneously odd or simultaneously even. Then, the following theorem is on the parity of the vertex set and edge set of G.

Proposition 2.8 *Let f be an IASSL of a given graph G, with respect to a ground set X. Then, if $V(G)$ and $E(G)$ are of same parity, then κ is an odd integer and if $V(G)$ and $E(G)$ are of different parity, then κ is an even integer, where κ is the number of subsets of X which are the set-labels of both vertices and edges.*

Proof Let f be a integer additive set-sequential labeling of a given graph G. Then, $f^*(G) = \mathcal{P}(X) - \{\emptyset\}$. Therefore, $|f^*(G)| = 2^{|X|} - 1$, which is an odd integer.

Case 1. Let $V(G)$ and $E(G)$ are of same parity. Then, $|V| + |E|$ is an even integer. Then, by Proposition 2.7, $2^{|X|} - 1 - \kappa$ is an even integer, which is possible only when κ is an odd integer.

Case 2. Let $V(G)$ and $E(G)$ are of different parity. Then, $|V| + |E|$ is an odd integer. Then, by Proposition 2.7, $2^{|X|} - 1 - \kappa$ is an odd integer, which is possible only when κ is an even integer. □

A relation between integer additive set-graceful labeling and an integer additive set-sequential labeling of a graph is established in the following result.

The following result determines the minimum number of vertices in a graph that admits an IASSL with respect to a finite non-empty set X.

Theorem 2.9 *Let X be a non-empty finite set of non-negative integers. Then, a graph G that admits an IASSL with respect to X have at least ρ vertices, where ρ is the number of elements in $\mathcal{P}(X)$ which are not the sum sets of any two elements of $\mathcal{P}(X)$.*

Proof Let f be an IASSL of a given graph G, with respect to a given ground set X. Let $\mathcal{A}$ be the collection of subsets of X such that no element in $\mathcal{A}$ is the sum sets any two subsets of X. Since f an IASL of G, all edge of G must have the set-labels which are the sum sets of the set-labels of their

end vertices. Hence, no element in $\mathcal{A}$ can be the set-label of any edge of G. But, since f is an IASSL of G, $\mathcal{A} \subset f^*(G) = f(V(G)) \cup f^+(E(G))$. Therefore, the minimum number of vertices of G is equal to the number of elements in the set $\mathcal{A}$. □

The structural properties of graphs which admit IASSLs arouse much interests. In the example of IASS-graphs, given in Figure 1, the graph G has some pendant vertices. Hence, there arises following questions in this context. Do an IASS-graph necessarily have pendant vertices? If so, what is the number of pendant vertices required for a graph G to admit an IASSL? Let us now proceed to find the solutions to these problems.

The minimum number of pendant vertices required in a given IASS-graph is explained in the following Theorem.

Theorem 2.10 *Let G admits an IASSL with respect to a ground set X and let $\mathcal{B}$ be the collection of subsets of X which are neither the sum sets of any two subsets of X nor their sum sets are subsets of X. If $\mathcal{B}$ is non-empty, then*

(1) $\{0\}$ *is the set-label of a vertex in G;*
(2) *the minimum number pendant vertices in G is cardinality of $\mathcal{B}$.*

Remark 2.11 Since the ground set X of an IASS-graph must contain the element 0, every subset A_i of X sum set of $\{0\}$ and A_i itself. In this sense, each subset A_i may be considered as a *trivial sum set* of two subsets of X.

In the following discussions, by a sum set of subsets of X, we mean the non-trivial sum sets of subsets of X.

Proof Let f be an IASSL of G with respect to a ground set X. Also, let $\mathcal{B}$ be the collection of subsets of X which are neither the sum sets of any two subsets of X nor their sum sets are subsets of X. Let $A \subset X$ be an element of $\mathcal{B}$. then A must be the set-label of a vertex of G. Since $A \in \mathcal{B}$, the only set that can be adjacent to A is $\{0\}$. Therefore, since G is a connected graph, $\{0\}$ must be the set-label of a vertex of G. More over, since A is an arbitrary vertex in $\mathcal{B}$, the minimum number of pendant vertices in G is $|\mathcal{B}|$. □

The following result thus establishes the existence of pendant vertices in an IASS-graph.

Theorem 2.12 *Every graph that admits an IASSL, with respect to a non-empty finite ground set X, have at least one pendant vertex.*

Proof Let the graph G admits an IASSL f with respect to a ground set X. Let $\mathcal{B}$ be the collection of subsets of X which are neither the sum sets of any two subsets of X nor their sum sets are subsets of X.

We claim that $\mathcal{B}$ is non-empty, which can be proved as follows. Since X is a finite set of non-negative integers, X has a smallest element, say x_1, and a greatest element x_l. Then, the subset $\{x_1, x_l\}$ belongs to $f^*(G)$. Since it is not the sum set any sets and is not a summand of any set in $\mathcal{P}(X)$, $\{x_1, x_l\} \in \mathcal{B}$. Therefore, $\mathcal{B}$ is non-empty.

Since $\mathcal{B}$ is non-empty, by Theorem 2.10, G has some pendant vertices. □

Remark 2.13 In view of the above results, we can make the following observations.

(1) No cycle C_n can have an IASSL;
(2) For $n \geqslant 2$, no complete graph K_n admits an IASSL.

(3) No complete bipartite graph $K_{m,n}$ admits an IASL.

The following result establish the existence of a graph that admits an IASSL with respect to a given ground set X.

Theorem 2.14 *For any non-empty finite set X of non-negative integers containing* 0*, there exists a graph G which admits an IASSL with respect to X.*

Proof Let X be a given non-empty finite set containing the element 0 and let $\mathcal{A} = \{A_i\}$, be the collection of subsets of X which are not the sum sets of any two subsets of X. Then, the set $\mathcal{A}' = \mathcal{P}(X) - \mathcal{A} \cup \{\emptyset\}$ is the set of all subsets of X which are the sum sets of any two subsets of X and hence the sum sets of two elements in $\mathcal{A}$.

What we need here is to construct a graph which admits an IASSL with respect to X. For this, begin with a vertex v_1. Label the vertex v_1 by the set $A_1 = \{0\}$. For $1 \leqslant i \leqslant |\mathcal{A}|$, create a new vertex v_i corresponding to each element in $\mathcal{A}$ and label v_i by the set $A_i \in \mathcal{A}$. Then, connect each of these vertices to V_1 as these vertices v_i can be adjacent only to the vertex v_1. Now that all elements in $\mathcal{A}$ are the set-labels of vertices of G, it remains the elements of $\mathcal{A}'$ for labeling the elements of G. For any $A'_r \in \mathcal{A}'$, we have $A'_r = A_i + A_j$, where $A_i, A_j \in \mathcal{A}$. Then, draw an edge e_r between v_i and v_j so that e_r has the set-label A'_r. This process can be repeated until all the elements in $\mathcal{A}'$ are also used for labeling the elements of G. Then, the resultant graph is an IASS-graph with respect to the ground set X. □

Figure 1 illustrates the existence of an IASSL for a given graph G.

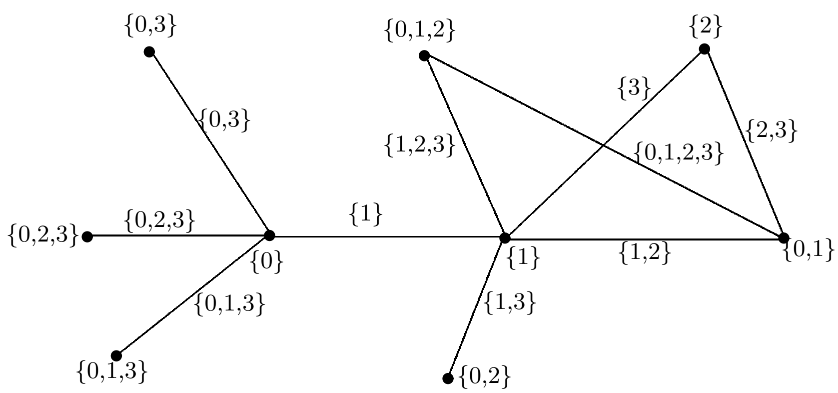


Figure 1

On the other hand, for a given graph G, the choice of a ground set X is also very important to have an integer additive set-sequential labeling. There are certain other restrictions in assigning set-labels to the elements of G. We explore the properties of a graph G that admits an IASSL with respect to a given ground set X. As a result, we have the following observations.

Proposition 2.15 *Let G be a connected integer additive set-sequential graph with respect to a ground set X. Let x_1 and x_2 be the two minimal non-zero elements of X. Then, no edges of G can have the set-labels $\{x_1\}$ and $\{x_2\}$.*

Proof In any IASL-graph G, the set-label of an edge is the sum set of the set-labels of its end vertices. Therefore, a subset A of the ground set X, that is not a sum set of any two subsets of X, can

not be the set-label of any edge of G. Since x_1 and x_2 are the minimal non-zero elements of X, $\{x_1\}$ and $\{x_2\}$ can not be the set-labels of any edge of G. □

Proposition 2.16 *Let G be a connected integer additive set-sequential graph with respect to a ground set X. Then, any subset A of X that contains the maximal element of X can be the set-label of a vertex v of G if and only if v is a pendant vertex that is adjacent to the vertex u having the set-label $\{0\}$.*

Proof Let x_n be the maximal element in X and let A be a subset of X that contains the element x_n. If possible, let A be the set-label of a vertex , say v, in G. Since G is a connected graph, there exists at least one vertex in G that is adjacent to v. Let u be an adjacent vertex of v in G and let B be its set-label. Then, the edge uv has the set-label $A+B$. If $B \neq \{0\}$, then there exists at least one element $x_i \neq 0$ in B and hence $x_i + x_n \notin X$ and hence not in $A+B$, which is a contradiction to the fact that G is an IASS-graph. □

Let us now discuss whether trees admit integer additive set-sequential labeling, with respect to a given ground set X.

Theorem 2.17 *A tree G admits an IASSL f with respect to a finite ground set X, then G has $2^{|X|-1}$ vertices.*

Proof Let G be a tree on n vertices. If possible, let G admits an IASSI. Then, $|E(G)| = n-1$. Therefore, $|V(G)|+|E(G)| = n+n-1 = 2n-1$. But, by Theorem 2.9, $2^{|X|}-1 = 2n-1 \implies n = 2^{|X|-1}$. □

Invoking the above results, we arrive at the following conclusion.

Theorem 2.18 *No connected graph G admits an integer additive set-sequential indexer.*

Proof Let G be a connected graph which admits an IASI f. By Proposition 2.4, if the induced function f^* is injective, then $\{0\}$ can not be the set-label of any element of G. But, by Propositions 2.15 and 2.16, every connected IASS-graph has a vertex with the set-label $\{0\}$. Hence, a connected graph G can not have an IASSI. □

The problem of characterizing (disconnected) graphs that admit IASSIs is relevant and interesting in this situation. Hence, we have

Theorem 2.19 *A graph G admits an integer additive set-sequential indexer f with respect to a ground set X if and only if G has ρ' isolated vertices, where ρ' is the number of subsets of X which are neither the sum sets of any two subsets of X nor the summands of any subsets of X.*

Proof Let f be an IASI defined on G, with respect to a ground set X. Let $\mathcal{B}$ be the collection of subsets of X which are neither the sum sets of any two subsets of X nor the summands of any subsets of X.

Assume that f is an IASSI of G. Then, the induced function f^* is an injective function. We have already showed that $\mathcal{B}$ is a non-empty set. By Theorem 2.10, $\{0\}$ must be the set-label of one vertex v in G and the vertices of G with set-labels from $\mathcal{B}$ can be adjacent only to the vertex v. By Remark 2.5, v must be an isolated vertex in G. Also note that $\{0\}$ is lso an element in $\mathcal{B}$. Therefore, all the vertices which have set-labels from $\mathcal{B}$ must also be isolated vertices of G. Hence G has $\rho' = |\mathcal{B}|$ isolated vertices.

Conversely, assume that G has $\rho' = |\mathcal{B}|$ isolated vertices. Then, label the isolated vertices of G by

the sets in $\mathcal{B}$ in an injective manner. Now, label the other vertices of G in an injective manner by other non-empty subsets of X which are not the sum sets of subsets of X in such a way that the subsets of X which are the sum sets of subsets of X are the set-labels of the edges of G. Clearly, this labeling is an IASSI of G. □

Analogous to Theorem 2.14, we can also establish the existence of an IASSI-graph with respect to a given non-empty ground set X.

Theorem 2.20 *For any non-empty finite set X of non-negative integers, there exists a graph G which admits an IASSI with respect to X.*

Figure 2 illustrates the existence of an IASSL for a given graph with isolated vertices.

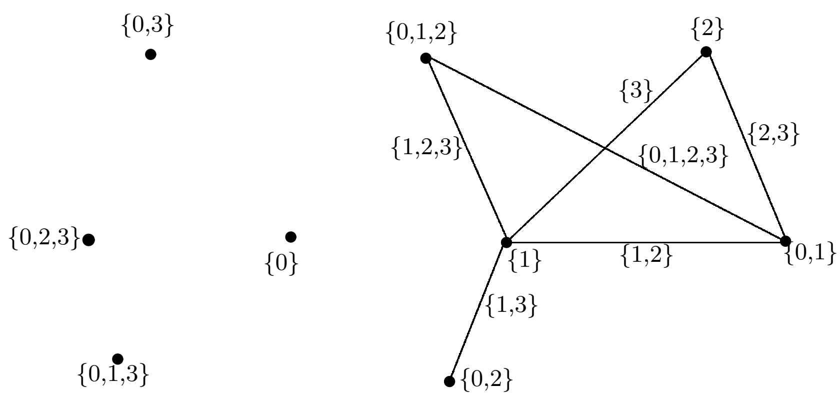


Figure 2

§3. Conclusion

In this paper, we have discussed an extension of set-sequential labeling of graphs to sum-set labelings and have studied the properties of certain graphs that admit IASSLs. Certain problems regarding the complete characterization of IASSI-graphs are still open.

We note that the admissibility of integer additive set-indexers by the graphs depends upon the nature of elements in X. A graph may admit an IASSL for some ground sets and may not admit an IASSL for some other ground sets. Hence, choosing a ground set is very important to discuss about IASSI-graphs.

There are several problems in this area which are promising for further studies. Characterization of different graph classes which admit integer additive set-sequential labelings and verification of the existence of integer additive set-sequential labelings for different graph operations, graph products and graph products are some of them. The integer additive set-indexers under which the vertices of a given graph are labeled by different standard sequences of non-negative integers, are also worth studying.

References

[1] B.D.Acharya, Set-valuations and their applications, *MRI Lecture notes in Applied Mathematics*, The Mehta Research Institute of Mathematics and Mathematical Physics, Allahabad,1983.

[2] B.D.Acharya, K.A.Germina, K.Abhishek and P.J.Slater, (2012). Some new results on set-graceful

and set-sequential graphs, *Journal of Combinatorics*, Information and System Sciences, 37(2-4), 145-155.

[3] B.D.Acharya and S.M.Hegde, Set-Sequential Graphs, *National Academy Science Letters*, 8(12)(1985), 387-390.

[4] J.A.Bondy and U.S.R.Murty, *Graph Theory*, Springer, 2008.

[5] A.Brandstädt, V.B.Le and J.P.Spinard, *Graph Classes:A Survey*, SIAM, Philadelphia, (1999).

[6] J.A.Gallian, A dynamic survey of graph labelling, The *Electronic Journal of Combinatorics*, (DS-6), (2013).

[7] K.A.Germina and T.M.K.Anandavally, Integer additive set-indexers of a graph: sum square graphs, *Journal of Combinatorics, Information and System Sciences*, 37(2-4)(2012), 345-358.

[8] K.A.Germina and N.K.Sudev, On weakly uniform integer additive set-indexers of graphs, *International Mathematical Forum*, 8(37)(2013), 1827-1834.

[9] F.Harary, *Graph Theory*, Addison-Wesley Publishing Company Inc., 1969.

[10] A.Rosa, On certain valuation of the vertices of a graph, In *Theory of Graphs*, Gordon and Breach, (1967).

[11] N.K.Sudev and K.A.Germina, On integer additive set-indexers of graphs, *International Journal of Mathematical Sciences & Engineering Applications*, 8(2)(2014), 11-22.

[12] N.K.Sudev and K.A.Germina, Some new results on strong integer additive set-indexers of graphs, *Discrete Mathematics, Algorithms & Applications*, 7(1)(2015), 1-11.

[13] N.K.Sudev and K.A.Germina, A study on integer additive set-graceful labelings of graphs, to appear.

[14] N.K.Sudev, K.A.Germina and K.P Chithra, A creative review on integer additive set-valued graphs, *International Journal of Scientific and Engineering Research*, 6(3)(2015), 372-378.

[15] D. B. West, *An Introduction to Graph Theory*, Pearson Education, 2001.

In silence, in steadiness, in severe abstraction, let him hold by himself, add observation to observation, patient of neglect, patient of reproach , and bide his own time , happy enough if he can satisfy himself alone that the day he has seen something truly.

By Ralph Waldo Emerson, an American thinker.

Author Information

Submission: Papers only in electronic form are considered for possible publication. Papers prepared in formats, viz., .tex, .dvi, .pdf, or.ps may be submitted electronically to one member of the Editorial Board for consideration both in **International Journal of Mathematical Combinatorics** and **Mathematical Combinatorics (International Book Series)**. An effort is made to publish a paper duly recommended by a referee within a period of 3 months. Articles received are immediately put the referees/members of the Editorial Board for their opinion who generally pass on the same in six week's time or less. In case of clear recommendation for publication, the paper is accommodated in an issue to appear next. Each submitted paper is not returned, hence we advise the authors to keep a copy of their submitted papers for further processing.

Abstract: Authors are requested to provide an abstract of not more than 250 words, latest Mathematics Subject Classification of the American Mathematical Society, Keywords and phrases. Statements of Lemmas, Propositions and Theorems should be set in italics and references should be arranged in alphabetical order by the surname of the first author in the following style:

Books

[4]Linfan Mao, *Combinatorial Geometry with Applications to Field Theory*, InfoQuest Press, 2009.
[12]W.S.Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

Research papers

[6]Linfan Mao, Mathematics on non-mathematics - A combinatorial contribution, *International J.Math. Combin.*, Vol.3(2014), 1-34.
[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

Figures: Figures should be drawn by TEXCAD in text directly, or as EPS file. In addition, all figures and tables should be numbered and the appropriate space reserved in the text, with the insertion point clearly indicated.

Copyright: It is assumed that the submitted manuscript has not been published and will not be simultaneously submitted or published elsewhere. By submitting a manuscript, the authors agree that the copyright for their articles is transferred to the publisher, if and when, the paper is accepted for publication. The publisher cannot take the responsibility of any loss of manuscript. Therefore, authors are requested to maintain a copy at their end.

Proofs: One set of galley proofs of a paper will be sent to the author submitting the paper, unless requested otherwise, without the original manuscript, for corrections after the paper is accepted for publication on the basis of the recommendation of referees. Corrections should be restricted to typesetting errors. Authors are advised to check their proofs very carefully before return.

September 2015

Contents

An International Book Series on Mathematical Combinatorics

ISBN 978-1-59973-377-7